Why is Dual-Pivot Quicksort Fast?

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TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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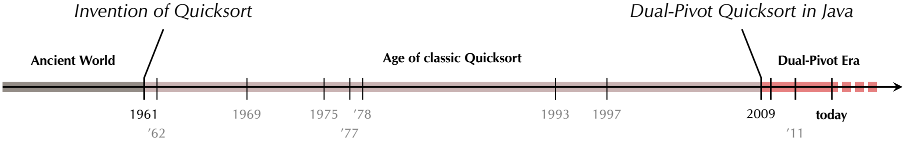
Theorietage 2015 Speyer

Sorting History



Sorting History

1961,62 **Hoare:** publication, first analysis

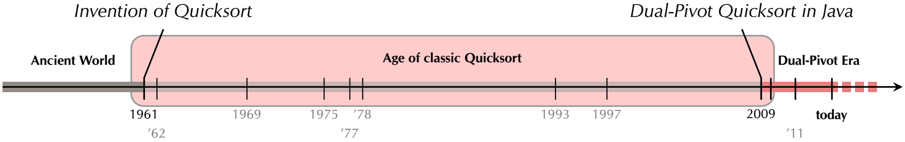
1969 **Singleton:** median-of-three & Insertionsort on small subarrays

1975-78 **Sedgewick:** analysis of many optimizations

1993 **Bentley, McIlroy:** duplicate elements & "ninther"

1997 **Musser:** $\mathcal{O}(n \log n)$ worst case by truncating recursion

$\rightsquigarrow$ Basic algorithm settled since 1961; latest tweaks from 1990's.
Since then: Almost identical in all programming libraries!



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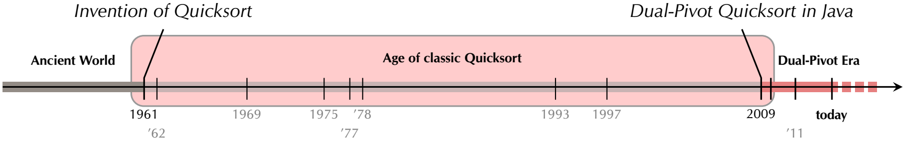
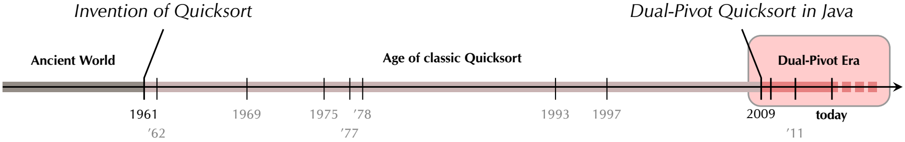


Sorting History

2008 – 2009	**Vladimir Yaroslavskiy** (developer at Sun) experiments with Quicksort with **two pivots**
11 Sep 2009	announcement on Java core library mailing list
29 Oct 2009	**inclusion** in development version of OpenJDK
2009 – 2011	optimizations by Joshua Bloch, Jon Bentley and others
28 July 2011	**public release** of Java 7 with Yaroslavskiy's Quicksort



Dual-Pivot Quicksort

Algorithm (Conceptual View)

1. Choose **two pivots** $P \leqslant Q$
2. For each element x, determine its **class**
    - **small** for $x < P$
    - **medium** for $P < x < Q$
    - **large** for $Q < x$

   by comparing x to **pivots** P and Q
3. Arrange elements according to classes:

4. Sort subarrays recursively.

How to implement 3 efficiently on arrays?

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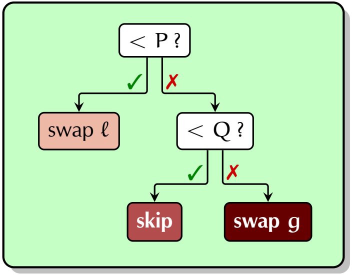

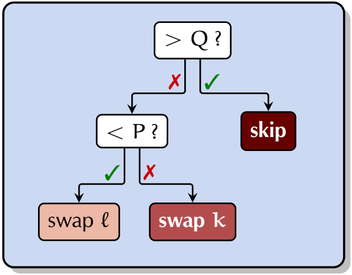

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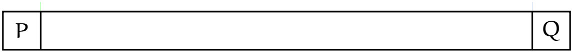

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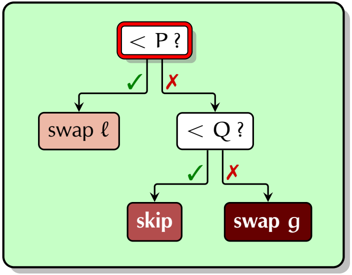

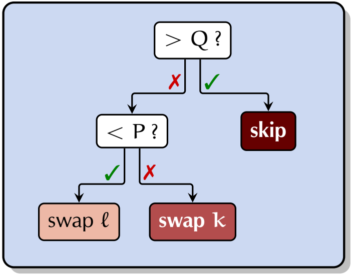

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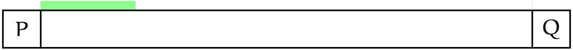

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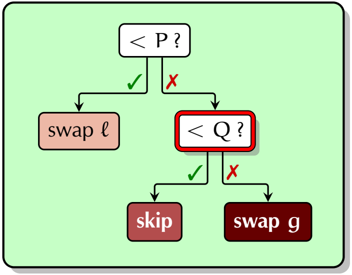

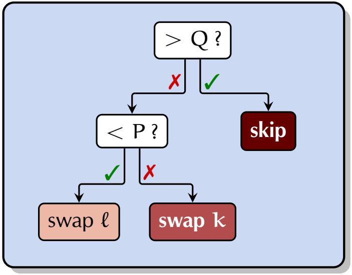

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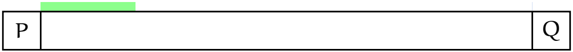

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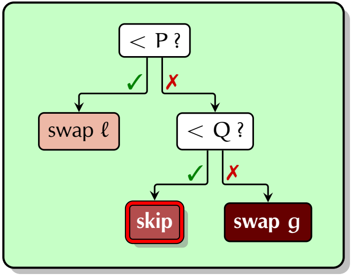

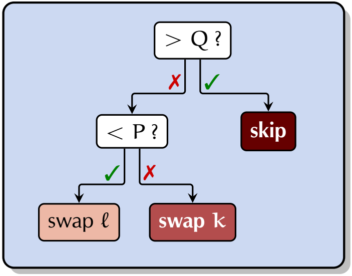

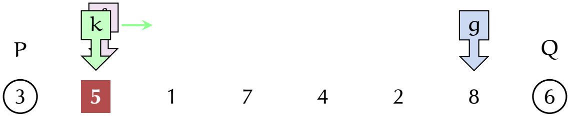

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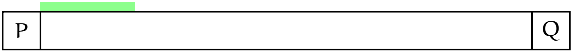

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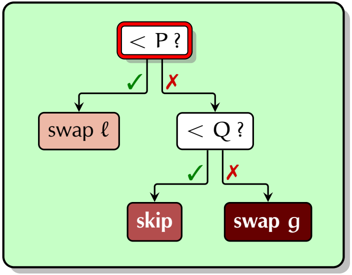

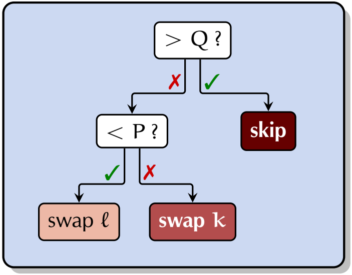

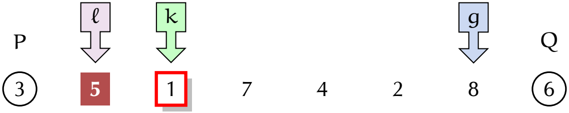

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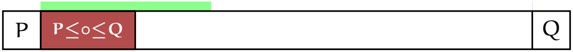

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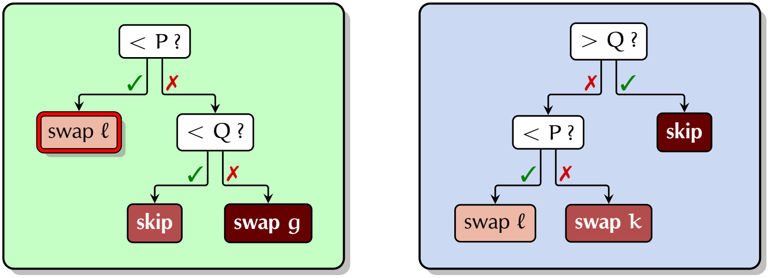

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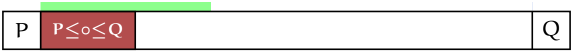

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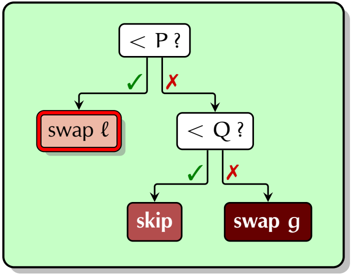

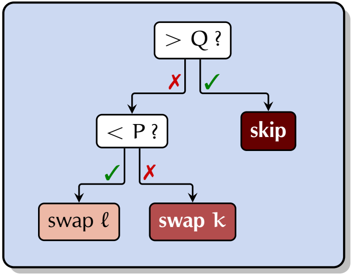

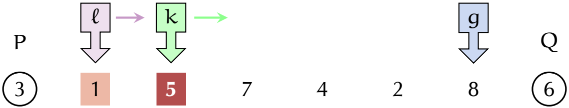

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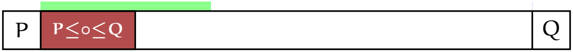

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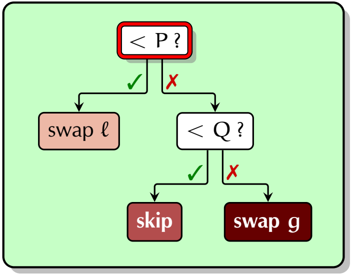

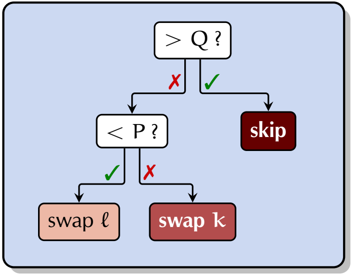

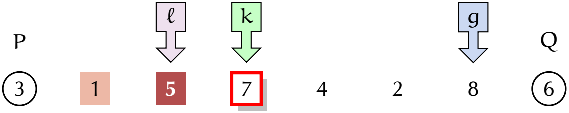

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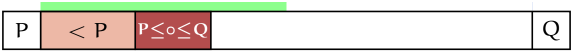

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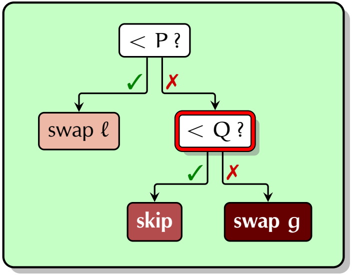

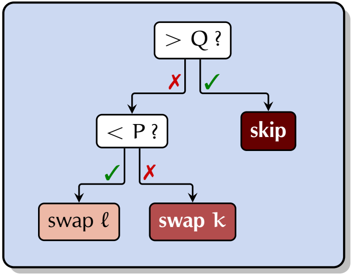

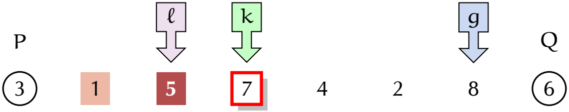

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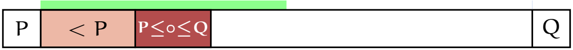

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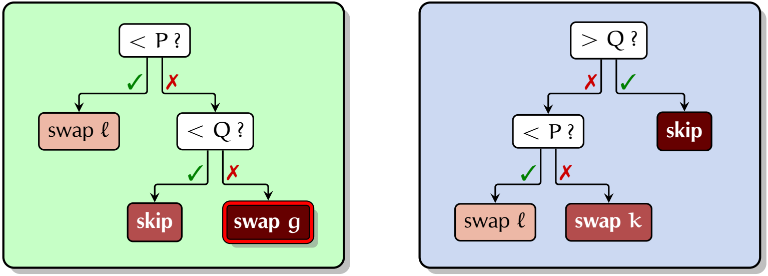

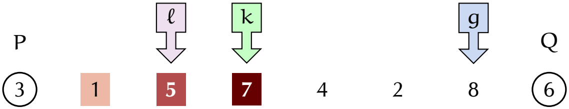

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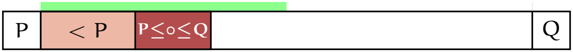

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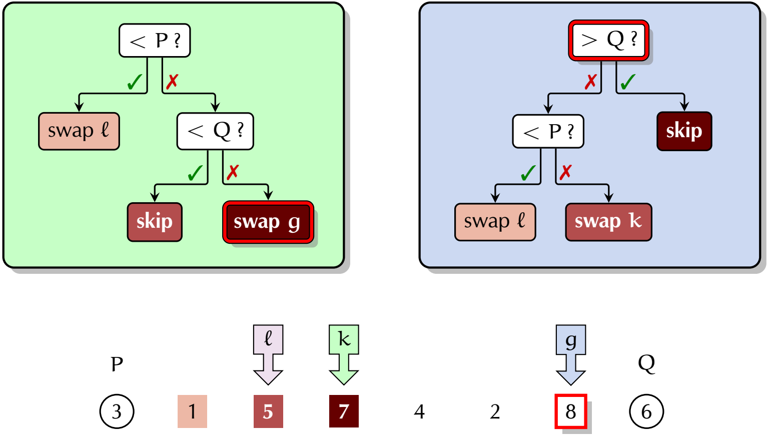

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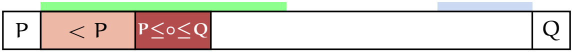

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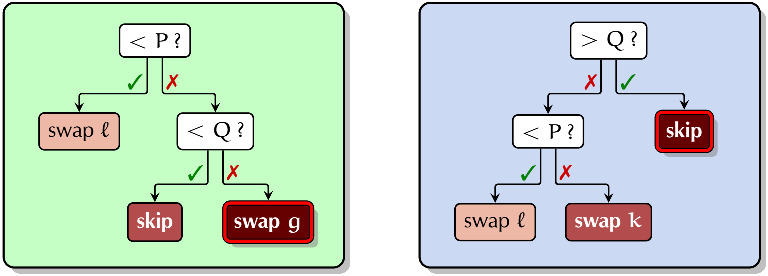

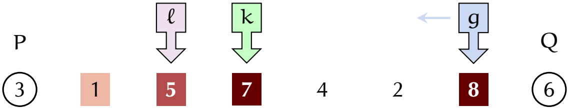

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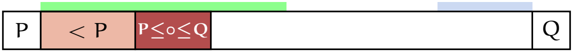

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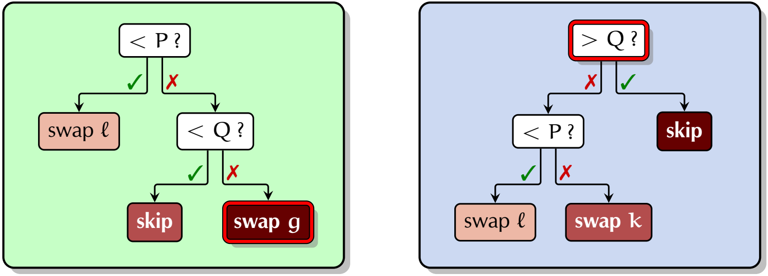

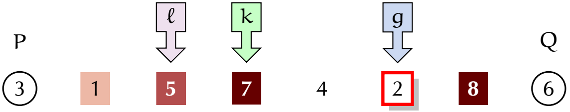

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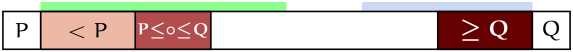

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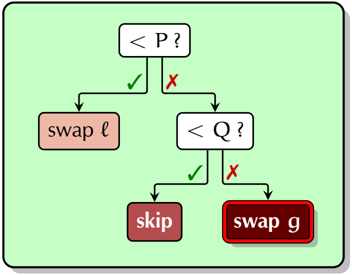

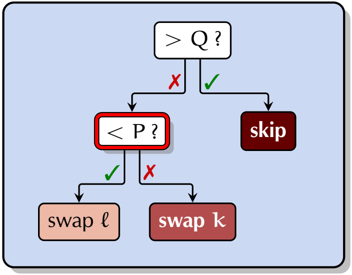

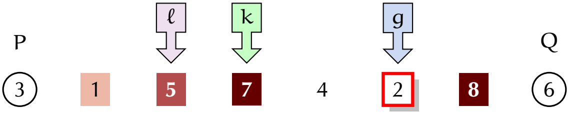

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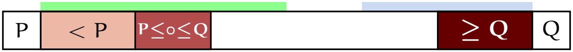

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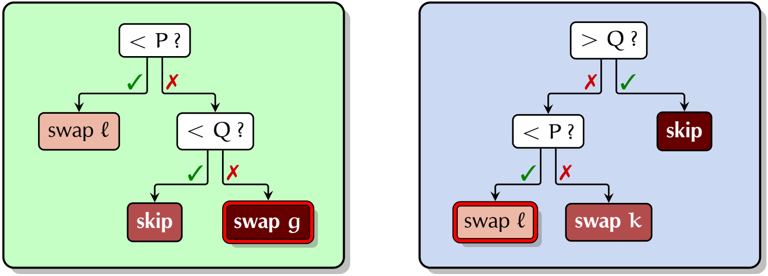

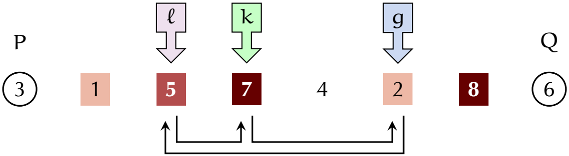

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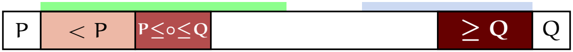

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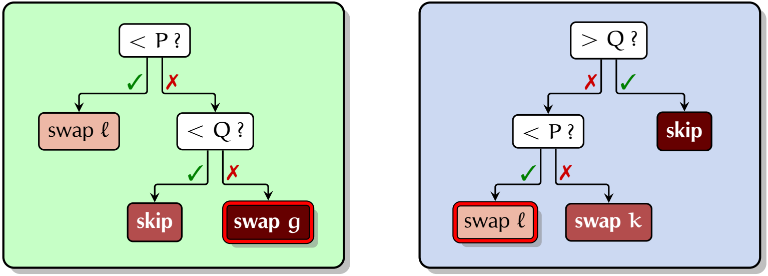

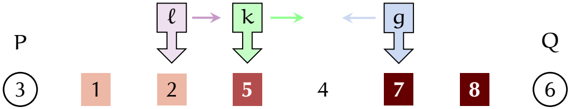

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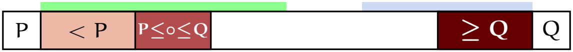

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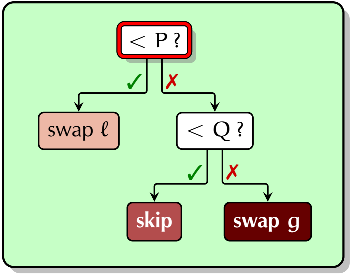

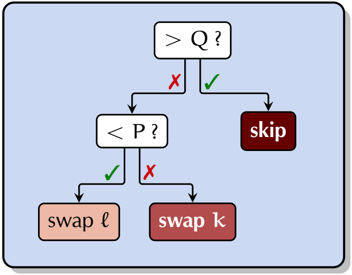

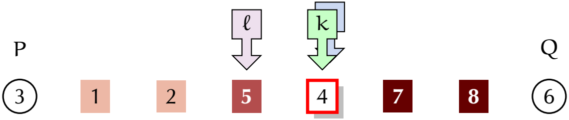

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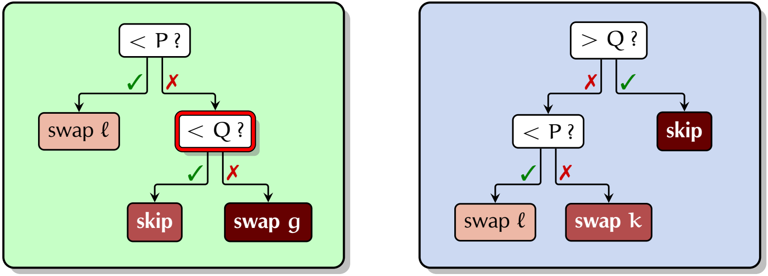

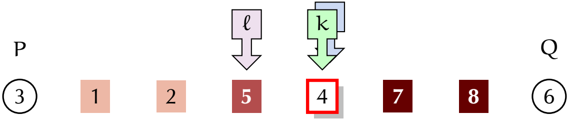

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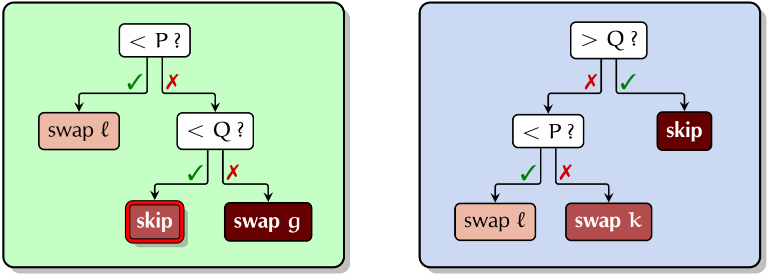

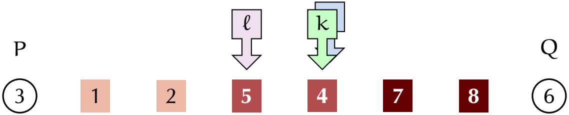

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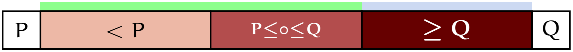

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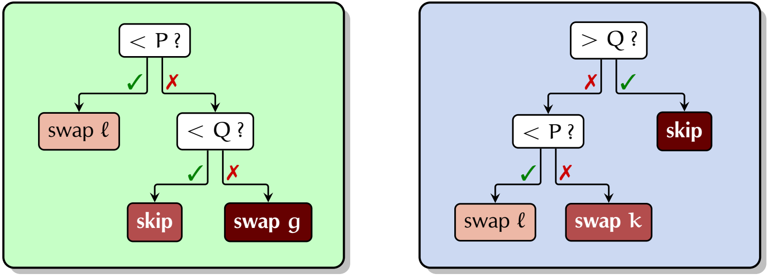

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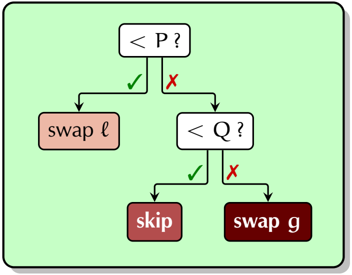

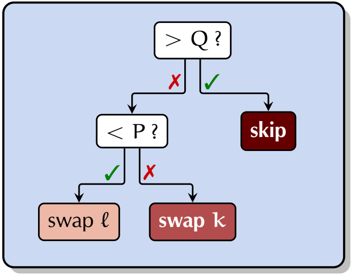

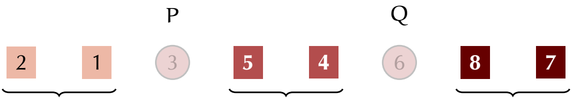

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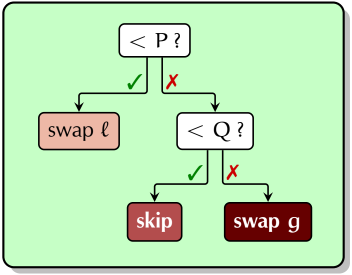
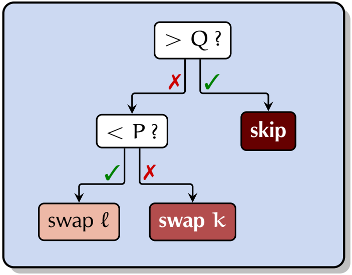
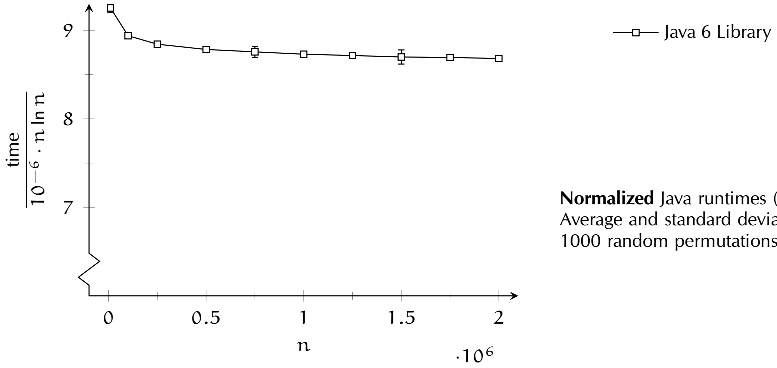




Running Time Experiments

Why switch to new, unknown algorithm?



Normalized Java runtimes (in *ms*).
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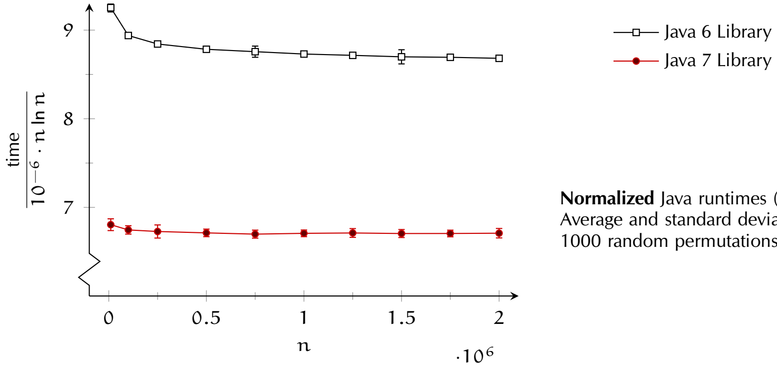

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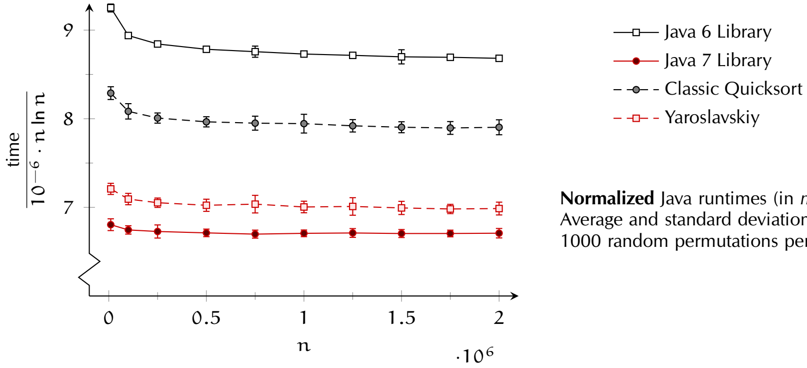

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- remains true for **basic** variants of algorithms: vs. !

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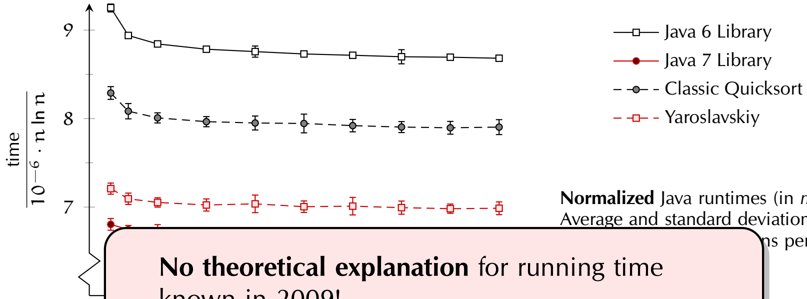

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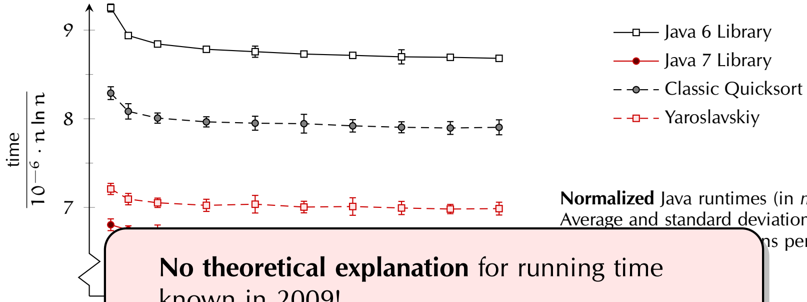

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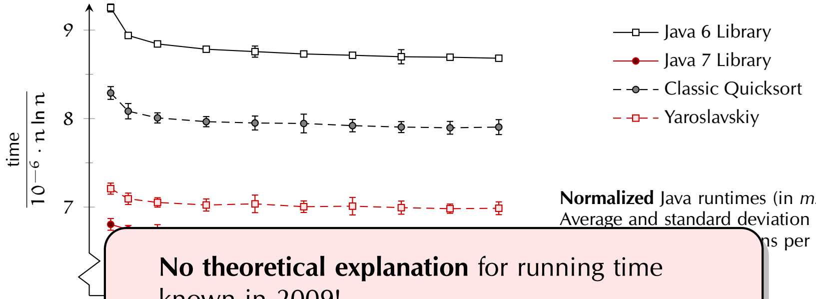

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Why did noone come up with this earlier?

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Track Record of Dual-Pivot Quicksort

- **Observation in practice:**
  Yaroslavskiy's Quicksort (YQS) **faster** than *classic Quicksort (CQS)* … **why?**

$\rightsquigarrow$ We did a mathematical analysis of YQS.

- **Traditional** cost measures do **not** explain observation!

	CQS	YQS	Relative
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Comparisons	2	1.9	-5%
Swaps	$0.\overline{3}$	0.6	$+80\%$
Bytecode Instructions	18	21.7	$+20.6\%$
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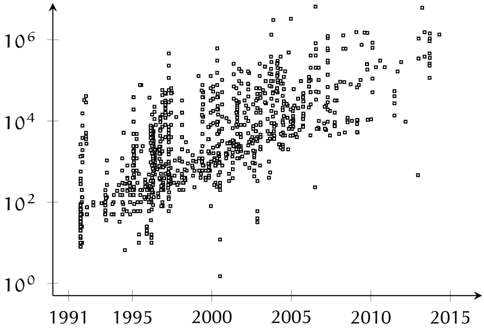

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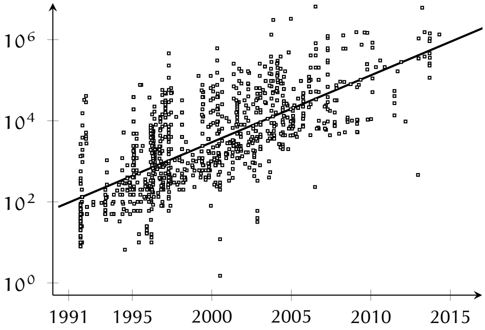

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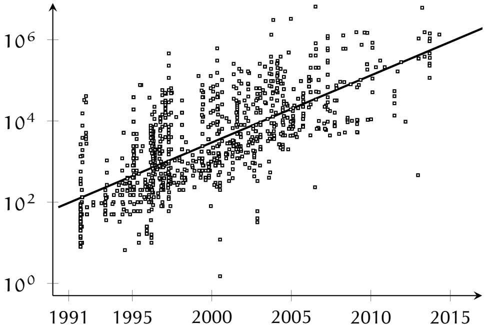

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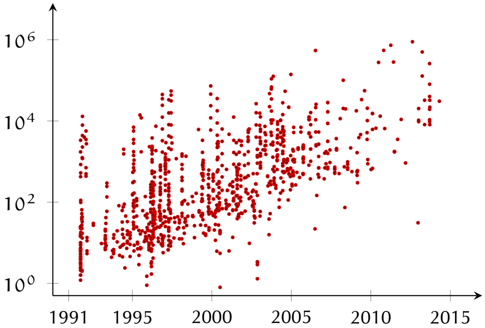

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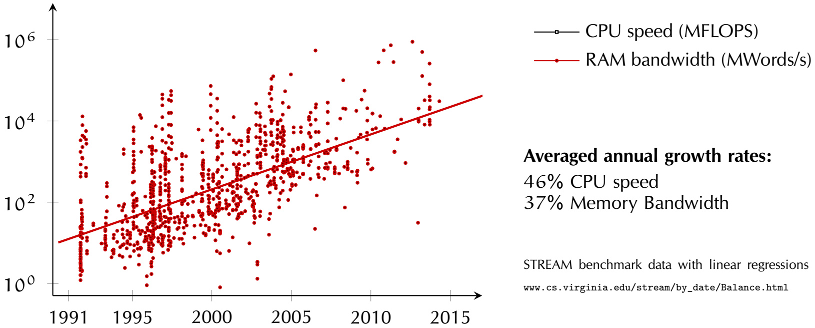

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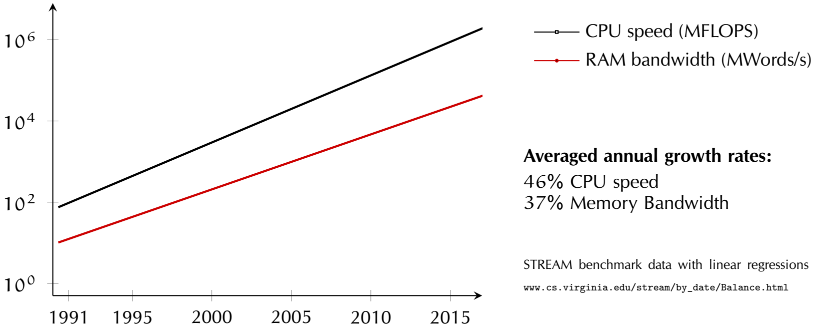

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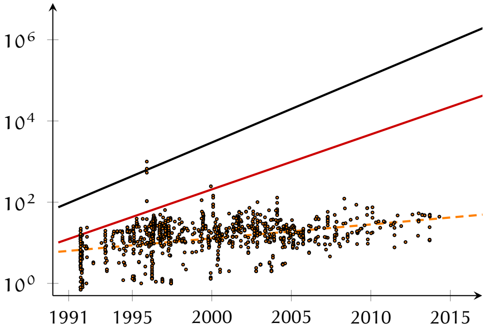

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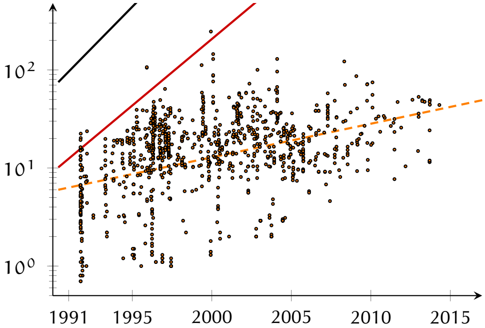

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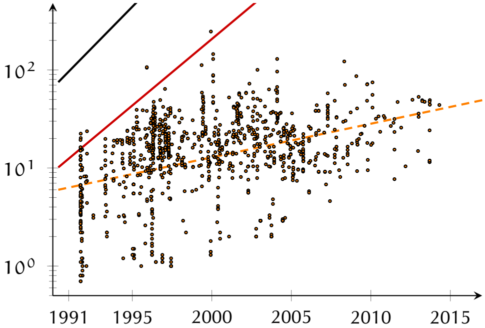

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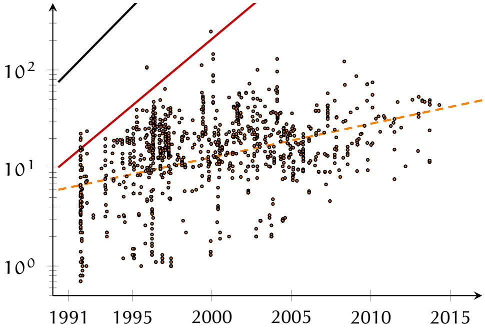

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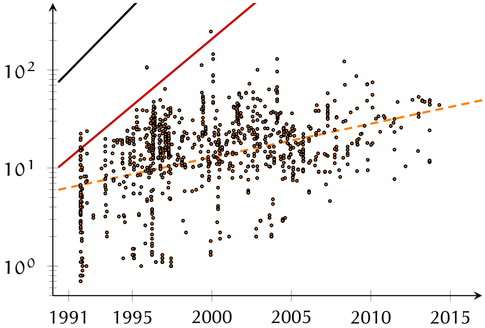

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- should **only** count memory accesses that are probably **not cached**

⇝ neither swaps nor comparisons are suitable measures!

My proposal: number of ***"scanned elements"***

- Machine model: Access to array only through iterators
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  - head left or right (one-directional)
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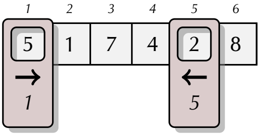

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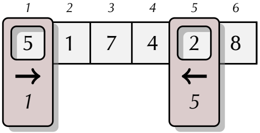

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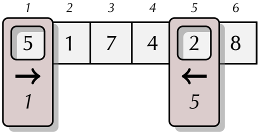

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Scanned Elements in CQS and YQS

How many scanned elements (SE) do we need for partitioning?

Classic Quicksort

- array scanned exactly once

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Yaroslavskiy's Quicksort

How does this translate to sorting costs?

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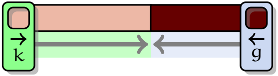

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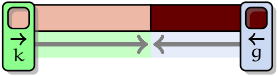

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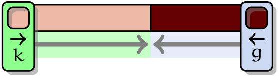

- array scanned exactly once

⇝ n scanned elements

Yaroslavskiy's Quicksort

⇝ $1.3n$ SE on average

How does this translate to sorting costs?

Classic Quicksort

Yaroslavskiy's Quicksort

Scanned Elements in CQS and YQS

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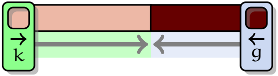

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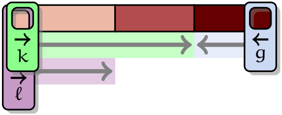

⇝ $1.\overline{3}n$ SE on average
worse than CQS!

How does this translate to sorting costs?

Classic Quicksort

⇝ $2\, n \ln n$ SE overall

Yaroslavskiy's Quicksort

⇝ $1.6\, n \ln n$ SE total

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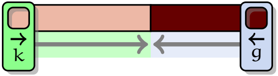

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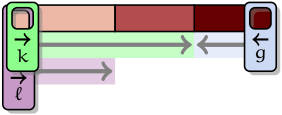

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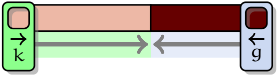

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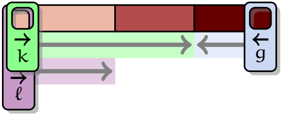

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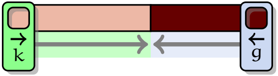

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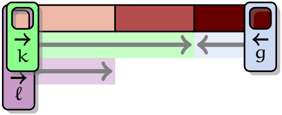

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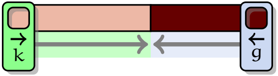

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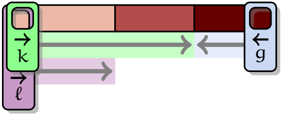

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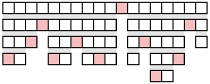

⇝ **2** $n \ln n$ SE overall

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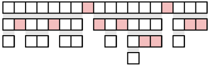

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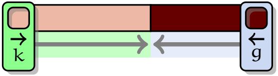

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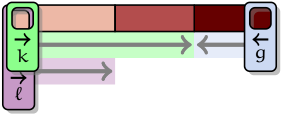

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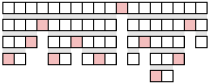

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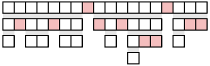

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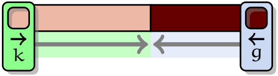

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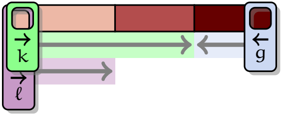

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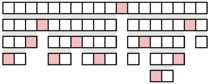

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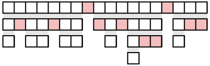

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- **Observation in practice:**
  Yaroslavskiy's Quicksort (YQS) **faster** than *classic Quicksort (CQS)* … **why?**

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- **Traditional** cost measures do **not** explain observation!

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Running Time	(from various experiments)		−10±2%
Comparisons	2	1.9	−5%
Swaps	$0.\overline{3}$	0.6	+80%
Bytecode Instructions	18	21.7	+20.6%
MMIX oops υ	11	13.1	+19.1%
MMIX mems μ	$2.\overline{6}$	2.8	+5%
Scanned Elements	2	1.6	−20%
Branch Mispredictions	0.57	0.58	+2%

$\cdot n \ln n + O(n)$, *average case results*

Only plausible explanation for running time: 20% less memory transfers in YQS.

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