Random induced subgraphs of Cayley graphs induced by transpositions

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1. Introduction

One central problem arising in parallel computing is to determine an optimal linkage of a given collection of processors. A particular class of processor linkages with point-to-point communication links are static interconnection networks. The latter are widely used for message-passing architectures. A static interconnection network can be represented as a graph. The binary n-cubes, Q_2^n, [1, 36] are a particularly well-studied class of interconnection networks [15, 20, 21, 41].

Akers *et al.* [2] observed the deficiencies of n-cubes as models for interconnection networks and proposed an alternative: the Cayley graph of the permutation group induced by the $(n-1)$ star-transpositions $(1\,i)$, which was denoted by $\Gamma(S_n, P_n)$. Pak [37] studied minimal decompositions of a particular permutation via star-transpositions and Irving *et al.* [30] extended his results. The star-graph $\Gamma(S_n, P_n)$ is in many aspects superior to n-cubes [1, 36]. Some properties of star-graphs studied in [26, 28, 29, 27, 31, 34] were cycle-embeddings and path-embeddings. Diameter and fault diameter of star-graphs were computed by Akers *et al.* [2, 33, 40] and Lin *et al.* [35] analyzed diagnosability. An alternative to n-cubes as interconnection networks are the bubble-sort graphs [3], studied by Tchuente [42]. The bubble-sort graph is the Cayley graph of the permutation group induced by all $n-1$ canonical transpositions $(i\ i+1)$, denoted by $\Gamma(S_n, B_n)$.

Recently, Araki [5] brought the attention to a generalization of star- and bubble-sort graphs, the Cayley graph generated by all transpositions [12]. The latter has direct connections to a problem

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of interest in computational biology: the evolutionary distances between species based on their genome order in the Cayley graph of signed permutations generated by reversals. A reversal is a special permutation that acts by flipping the order as well as the signs of a segment of genes. Hannenhalli and Pevzner [23] presented an algorithm computing minimal number of reversals needed to transform one sequence of distinct genes into a given signed permutation. For distant genomes, however, it is well-known, that the true evolutionary distance is generally much greater than the shortest distance [43, 13, 11, 7]. In order to obtain a more realistic estimate of the true evolutionary distance, the expected reversal distance was shifted into focus. Its computation, however, has proved to be hard and motivated models better suited for computation. Point in case is the work of Eriksen *et al.* [19], where the authors derive a closed formula for the expected transposition distance and subsequently show how to use it as an approximation of the expected reversal distance. Berestycki and Durrett [8] studied the shortest distance of random walks over Cayley graphs generated by all transpositions and canonical transpositions, respectively, and compared the shortest distance with the expected distance [19].

The theory of random graphs was pioneered by Erdös and Rényi in the late 1950s [17, 18], who analyzed the phase transition of $G(n, p_n)$, the random graph containing n vertices in which an edge $\{i, j\}$ is selected with independent probability p_n. For $p_n = \frac{c}{n}$ and $c < 1$, the largest component in $G(n, p_n)$ is a.s. of size $O(\log n)$. For $p_n = \frac{1+\theta \cdot n^{-\frac{1}{3}}}{n}$, where $\theta > 0$, a.s. a largest component of size $O(n^{\frac{2}{3}})$ emerges. For $p_n = \frac{c}{n}$ and $c > 1$, we have a.s. a unique largest component of size $O(n)$ and all other components are smaller than $O(\log n)$. Erdös and Rényi's construction of the giant component [17, 18] has motivated Lemma 3, which assures the existence of certain subtrees of size $\lfloor \frac{1}{4} n^{\frac{2}{3}} \rfloor$. For a review of Erdös-Rényi random graph theory, see Durrett [16] or van der Hofstad [22].

In this paper we study a subgraph of the Cayley graph generated by all transpositions, the Cayley graph $\Gamma(S_n, T_n)$, where T_n is a minimal generating set of transpositions. Setting $T_n = P_n$ and $T_n = B_n$ we can recover the star- and the bubble-sort graph as particular instances. We study structural properties of $\Gamma(S_n, T_n)$ in terms of the random graph obtained by selecting permutations with independent probability. The main result of this paper is

Theorem 1. *Let* $\lambda_n = \frac{1+\epsilon_n}{n-1}$*, where* $n^{-\frac{1}{3}+\delta} \le \epsilon_n < 1$ *and* $\delta > 0$*. Let* T_n *be a minimal generating set of transpositions and let* Γ_n *denote the random induced subgraph of* $\Gamma(S_n, T_n)$*, obtained by independently selecting each permutation with probability* λ_n*. Then* Γ_n *has a.s. a unique giant component,* $C_n^{(1)}$*, whose size is given by*

$$|C_n^{(1)}| = (1 + o(1)) \cdot x(\epsilon_n) \cdot \frac{1 + \epsilon_n}{n - 1} \cdot n!, \tag{1.1}$$

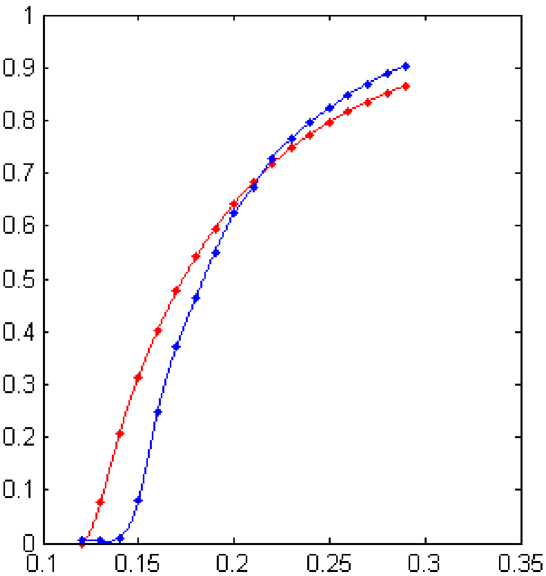


FIGURE 1. The evolution of the giant component in random induced subgraphs of $\Gamma(S_9, P_9)$. We display the relative size of the giant component $\frac{|C_9^{(1)}|}{|\Gamma_9|}$ as a function of $\lambda_9 = (1+\epsilon)/8$ as data-curve (blue) versus the growth predicted by Theorem 1 (red).

where $x(\epsilon_n) > 0$ *is the survival probability of a Poisson branching process with parameter* $\lambda = 1+\epsilon_n$ *and also the unique positive root of* $e^{-(1+\epsilon_n)y} = 1-y$. *Particularly, if* $n^{-\frac{1}{3}+\delta} \le \epsilon_n = o(1)$, *then we have* $x(\epsilon_n) = (2+o(1))\epsilon_n$.

In contrast to vertex-induced random graphs, edge-induced random graphs have been studied quite extensively. Random induced subgraphs of n-cubes [9, 38]. as well as $G(n, p_n)$ and random induced subgraphs of $\Gamma(S_n, T_n)$ exhibit a giant component for very small vertex selection probabilities. One might speculate that the critical probability $p_n = \frac{1+\theta\cdot n^{-\frac{1}{3}}}{n}$ is determined by the size of the generator set. Note that $|T_n| = n-1$ holds for any minimal generating set of transpositions and the size of the generator set for n-cube is n. Specific properties of n-cubes, like for instance, the isoperimetric inequality [24], do not play a key role for establishing the existence of the giant component. The isoperimetric inequality depends on an inductive argument using particular properties of a linear ordering of the vertices of an n-cube. This induction cannot be carried out for Cayley graphs over canonical transpositions. In this paper any argument involving (vertex) boundaries follows from a generic estimate of the vertex boundary in Cayley graphs due to Aldous [4, 6].

The paper is organized as follows: after introducing in Section 2 our notation and some basic facts about branching processes, we analyze in Section 3 vertices contained in polynomial size subcomponents. The strategy is similar to that in [38], where first a specific branching process is embedded (for its first $\lfloor \frac{1}{4} n^{\frac{2}{3}} \rfloor$ steps) into $\Gamma(S_n, T_n)$. It is its survival probability that provides a lower bound on the probability that a given vertex is contained in a subcomponent of arbitrary, polynomial size. In Section 4 we "sandwich" this bound by showing that there are many vertices in "small" components. Only here we use $\epsilon < 1$. In Section 5 we show that there are many vertex disjoint paths between certain splits of permutations. The a.s. existence of the giant component follows using the ideas of Ajtai *et al.* [1].

2. Background and notation

Let S_n denote the symmetric group over $[n]$. We write a permutation $\pi \in S_n$ as an n-tuple $(x_1, x_2, \cdots, x_n)$, i.e.,

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} = (x_1, x_2, \cdots, x_n).$$

Particularly we use $(i\, j)$ to briefly denote the transpositions that merely interchange the elements at positions i and j of the identity permutation. Plainly, we have

$$(x_1, \cdots, x_i, x_{i+1}, \cdots, x_{j-1}, x_j, \cdots, x_n) \cdot (i\, j) = (x_1, \cdots, x_j, x_{i+1}, \cdots, x_{j-1}, x_i, \cdots, x_n). \tag{2.1}$$

Furthermore, we set $((x_1, \cdots, x_n))_m = x_m$ i.e. extracting the m-th coordinate. Let $T_n \subset S_n$ be a minimal generating set of transpositions. We consider the Cayley graph $\Gamma(S_n, T_n)$, having vertex set S_n and edges $\{v, v'\}$ where $v^{-1} \cdot v' \in T_n$. For $v, v' \in S_n$, let $d(v, v')$ be the minimal number of T_n-transpositions by which v and v' differ. For $A \subset S_n$ we set $\mathsf{B}(A, j) = \{v \in S_n \mid \exists\, \alpha \in A;\, d(v, \alpha) \leq j\}$ and $\mathsf{d}(A, i) = \{v \in S_n \setminus A \mid \exists\, \alpha \in A;\, d(v, \alpha) = i\}$ and call $\mathsf{B}(A, j)$ and $\mathsf{d}(A) = \mathsf{d}(A, 1)$ the ball of radius j around A and the vertex boundary of A in $\Gamma(S_n, T_n)$. If $A = \{\alpha\}$ we simply write $\mathsf{B}(\alpha, j)$. Let $D, E \subset S_n$, we call D ℓ-dense in E if $\mathsf{B}(\sigma, \ell) \cap D \neq \varnothing$ for any $\sigma \in E$. Let "$\leq$" be the following linear order over $\Gamma(S_n, T_n)$

$$\sigma \leq \tau \quad \Longleftrightarrow \quad \sigma = \tau \text{ or } \sigma <_{\text{lex}} \tau, \tag{2.2}$$

where $<_{\rm lex}$ denotes the lexicographical order. Any notion of minimal or smallest element in a subset $A \in S_n$ refers to eq. (2.2).

Let $\Gamma_{\lambda_n}(S_n, T_n)$ be the probability space (random graph) consisting of $\Gamma(S_n, T_n)$-subgraphs, Γ_n, induced by selecting each $\Gamma(S_n, T_n)$-vertex with independent probability λ_n. A property M is a subset of induced subgraphs of $\Gamma(S_n, T_n)$ closed under graph isomorphisms. The terminology "M holds a.s." is equivalent to $\lim_{n\to\infty} \mathbb{P}(\mathsf{M}) = 1$. A component of Γ_n is a maximal, connected, induced Γ_n-subgraph, C_n. The largest Γ_n-component is denoted by $C_n^{(1)}$. We write $x_n \sim y_n$ if and only if (a) $\lim_{n\to\infty} x_n/y_n$ exists and (b) $\lim_{n\to\infty} x_n/y_n = 1$. We set $g(n) = o(f(n))$ if and only if $g(n)/f(n) \to 0$. A largest Γ_n-component $C_n^{(1)}$ is called giant if it is unique, i.e. any other component, C_n, satisfies $|C_n| = o(|C_n^{(1)}|)$.

We furthermore write $g(n) = O(f(n))$ as $n \to \infty$ if and only if $\frac{g(n)}{f(n)}$ is bounded as $n \to \infty$, i.e., for arbitrary $M > 0$, there exists a constant C (independent of M) such that for all $n > M$, $\left|\frac{g(n)}{f(n)}\right| \leq C$.

Let $Z_n = \sum_{i=1}^n \xi_i$ be a sum of mutually independent indicator random variables (r.v.), ξ_i having values in $\{0, 1\}$. Then we have, [14], for $\eta > 0$ and $c_\eta = \min\{-\ln(e^\eta[1+\eta]^{-[1+\eta]}), \frac{\eta^2}{2}\}$

$$\mathbb{P}(\,|\,Z_n - \mathbb{E}[Z_n]\,|\, > \eta\,\mathbb{E}[Z_n]\,) \leq 2e^{-c_\eta \mathbb{E}[Z_n]}\,. \tag{2.3}$$

In Lemma 3 we shall use

$$\mathbb{P}(\,Z_n\, < (1-\eta)\,\mathbb{E}[Z_n]\,) \leq e^{-\frac{\eta^2}{2}\cdot \mathbb{E}[Z_n]}\,. \tag{2.4}$$

In the following we shall assume that n is always sufficiently large. Let us next recall Chebyshev's inequality [39]: suppose ξ is a r.v. having finite variance, $\mathbb{V}(\xi)$, and $m > 0$. Then

$$\mathbb{P}(|\xi - \mathbb{E}(\xi)| \geq m) \leq \frac{\mathbb{V}(\xi)}{m^2}. \tag{2.5}$$

Furthermore, the r.v. X is $\mathrm{Bi}(n, \lambda_n)$-distributed if

$$\mathbb{P}(X = \ell) = \binom{n}{\ell} \lambda_n^\ell\,(1-\lambda_n)^{n-\ell}$$

and we call X binomially distributed (with parameters n, λ_n).

We next come to some basic facts about binomial branching processes, $\mathcal{P}_n = \mathcal{P}_n(p)$ [25, 32]. Suppose the process $\mathcal{P}_n$ is initialized at ξ. Let $(\xi_i^{(t)})$, $i, t \in \mathbb{N}$ count the number of "offspring" of the ith-individual of generation $(t-1)$ and in particular $\xi_1^{(1)}$ counts the number of offspring generated by ξ, in which all the r.v.s $\xi_i^{(t)}$ are $\mathrm{Bi}(n,p)$-distributed. Let $\mathcal{P}_0 = \mathcal{P}_0(p)$ denote the branching process for which $\xi_1^{(1)}$ is $\mathrm{Bi}(n,p)$- and all $\xi_i^{(t)} \neq \xi_1^{(1)}$ are $\mathrm{Bi}(n-1,p)$-distributed. Furthermore, let $\mathcal{P}_P(\lambda)$, $(\lambda > 0)$ denote the Poisson branching process in which all individuals $\xi_i^{(t)}$ generate offspring according to the Poisson distribution, i.e., $\mathbb{P}(\xi_i^{(t)} = j) = \frac{\lambda^j}{j!}e^{-\lambda}$. We accordingly consider the family of r.v. $(Z_i^x)_{i\in\mathbb{N}_0}$: $Z_0^x = 1$ and $Z_t^x = \sum_{i=1}^{Z_{t-1}^x} \xi_i^{(t)}$ for $t \geq 1$ and interpret Z_t^x as the number of individuals "alive" in generation t, where $x \in \{n, 0, P\}$. Of particular interest for us will be the limit $\lim_{t\to\infty} \mathbb{P}(Z_t^x > 0)$, i.e. the probability of infinite survival. We write

$$\pi_0(p) = \lim_{t\to\infty} \mathbb{P}(Z_t^0 > 0),\ \pi_n(p) = \lim_{t\to\infty} \mathbb{P}(Z_t^n > 0) \text{ and } \pi_P(\lambda) = \lim_{t\to\infty} \mathbb{P}(Z_t^P > 0)$$

for the survival probability of $\mathcal{P}_0$, $\mathcal{P}_n$ and $\mathcal{P}_P(\lambda)$, respectively.

Lemma 1. [10] *Let $p = \chi_n/n$ where $\chi_n > 1$, then $\pi_0(p) = (1+o(1))\pi_P(\chi_n)$, where $\pi_P(\chi_n) > 0$ is the unique positive root of the equation $e^{-\chi_n y} = 1 - y$. Particularly, if $\chi_n = 1 + \epsilon_n$ where $0 < \epsilon_n = o(1)$ and $s = o(n\epsilon_n)$,*

$$\pi_0(p) = (1+o(1))\pi_{n-s}(p) = (2+o(1))\epsilon_n.$$

Proof. Let $f_m(s)$ be the probability generating function for the binomial distribution $\mathrm{Bi}(m, \frac{\chi_n}{n})$ and $g_{\chi_n}(s)$ be the probability generating function for Poisson distribution with parameter $\lambda = \chi_n$, i.e.,

$$\begin{aligned} f_m(s) &= \sum_{j=1}^{m} P(\xi_i^{(t)} = j) \cdot s^j \\ &= \sum_{j=1}^{m} \binom{m}{j} (\frac{\chi_n s}{n})^j (1 - \frac{\chi_n}{n})^{m-j} \\ &= \left[1 - (1-s)\frac{\chi_n}{n}\right]^m \\ g_{\chi_n}(s) &= \sum_{i=0}^{\infty} e^{-\chi_n} \cdot \frac{(\chi_n)^i}{i!} \cdot s^i = e^{(s-1)\chi_n}. \end{aligned}$$

Then π_n and π_{χ_n}, the survival probabilities for the binomial distribution and Poisson distribution, are the roots of $f_n(1-s) = 1-s$ and $g_{\chi_n}(1-s) = 1-s$, respectively. Clearly, $f_n(1-s) =$

$g_{\chi_n}(1-s)e^{O(\frac{1}{n})}$, whence

$$\begin{aligned} f_n(1-\pi_{\chi_n}+o(1)) &= g_{\chi_n}(1-\pi_{\chi_n}+o(1))\cdot e^{O(\frac{1}{n})} \\ &= e^{-\pi_{\chi_n}\chi_n}e^{o(1)\chi_n+O(\frac{1}{n})} \\ &= e^{-\pi_{\chi_n}\chi_n}(1+o(1)) = 1-\pi_{\chi_n}+o(1). \end{aligned} \tag{2.6}$$

Since $E(\xi_i^{(t)}) = f_n'(1) = \frac{\chi_n}{n}n = \chi_n > 1$, where $\xi_i^{(t)}$ counts the number of "offspring" of the ith-individual of generation $(t-1)$, we can conclude that π_n is the unique positive root of $f_n(1-s) = 1-s$. In view of eq. (2.6) we have $\pi_n = \pi_{\chi_n} + o(1) = \pi_{\chi_n}(1+o(1))$. This implies

$$\pi_0(\frac{\chi_n}{n}) = (1+o(1))\pi_n = \pi_{\chi_n}(1+o(1)),$$

where $x = \pi_{\chi_n}$ is the unique positive root of $e^{-\chi_n\cdot x} = 1-x$. In case of $0<\epsilon_n = o(1)$, we can compute π_n explicitly via the binomial branching process $\mathcal{P}_m(\frac{\chi_n}{n})$. To this end we consider the root of $f_{n-k}(1-s) = 1-s$ where $k = o(n\epsilon_n)$ and observe

$$\begin{aligned} \pi_n(\frac{1+\epsilon_n}{n}) &= \frac{2n\epsilon_n}{n-1} + O(\epsilon_n^2) = 2\epsilon_n + O(\frac{\epsilon_n}{n}) + O(\epsilon_n^2) = (2+o(1))\epsilon_n \\ \pi_{n-k}(\frac{1+\epsilon_n}{n}) &= 2\epsilon_n + O(\frac{\epsilon_n}{n}) + O(\frac{k}{n}) + O(\epsilon_n^2) = (2+o(1))\epsilon_n. \end{aligned}$$

Using $\pi_{n-k}(\frac{1+\epsilon_n}{n}) \le \pi_0(\frac{1+\epsilon_n}{n}) \le \pi_n(\frac{1+\epsilon_n}{n})$, we arrive at

$$\pi_0(\frac{1+\epsilon_n}{n}) = (1+o(1))\pi_n(\frac{1+\epsilon_n}{n}) = (1+o(1))(2+o(1))\epsilon_n = (2+o(1))\epsilon_n$$

and the lemma follows. □

3. Components of polynomial size

Let ϵ be a positive constant satisfying $0<\epsilon<1$. Suppose $y = x > 0$ is the unique positive root of $\exp(-(1+\epsilon)y) = 1-y$ and

$$\wp(\epsilon_n) = \begin{cases} (1+o(1))x & \text{for } \epsilon_n = \epsilon > 0 \\ (2+o(1))\epsilon_n & \text{for } 0<\epsilon_n = o(1). \end{cases} \tag{3.1}$$

According to Lemma 1, $\wp(\epsilon_n) = \pi_0(\frac{1+\epsilon_n}{n-1})$ is the survival probability of branching process $\mathcal{P}_0(\frac{1+\epsilon_n}{n-1})$. For $k\in\mathbb{N}$ we set

$$\mu_n = \lfloor \frac{1}{2k(k+1)}n^{\frac{2}{3}}\rfloor, \quad \ell_n = \lfloor \frac{k}{2(k+1)}n^{\frac{2}{3}}\rfloor, \quad \text{and} \quad r_n = n - k\mu_n - \ell_n. \tag{3.2}$$

Without loss of generality we can assume $\mu_n, \ell_n, r_n \in \mathbb{N}$ and establish some basic properties of the Cayley graph $\Gamma(S_n, T_n)$:

Lemma 2. *Let T_n be a minimal generating set of S_n consisting of transpositions, then we have*
(1) *T_n has cardinality $n-1$ and corresponds uniquely to a labeled tree over $[n]$, denoted by $\mathcal{T}_n$.*
(2) *there exists a sequence $(v_i)_{2\le i}$ such that $T_n = \{(v_i\, s_i) \mid 2 \le i \le n\}$ and*

$$\forall\, j < i; \quad x_{v_i} = ((x_1, \dots, x_n) \cdot (v_j\, s_j))_{v_i} \quad \neq \quad ((x_1, \dots, x_n) \cdot (v_i\, s_i))_{v_i}. \tag{3.3}$$

(3) *the diameter of $\Gamma(S_n, T_n)$ is given by*

$$\mathrm{diam}(\Gamma(S_n, T_n)) \le \binom{n}{2}. \tag{3.4}$$

Proof. It is straightforward to prove by induction that $|T_n| = n-1$. We next consider the graph $\mathcal{T}_n$ over $[n]$, having edge-set T_n. Since $\langle T_n \rangle = S_n$, $\mathcal{T}_n$ is connected and since T_n is independent, $\mathcal{T}_n$ is a tree. This establishes the mapping

$$\psi\colon \{T_n \mid T_n \text{ is a maximal independent transposition set}\} \longrightarrow \{\mathcal{T}_n \mid \mathcal{T}_n \text{ is a tree over } [n]\}.$$

Furthermore, ψ has an inverse; as the edges of a tree over $[n]$ give rise to a maximal independent set of transpositions that generate S_n, whence assertion (1). Note that the critical probability $\lambda_n = \frac{1+\epsilon_n}{n-1}$ of Theorem 1 is determined by the cardinality of the generator set T_n, i.e., $|T_n| = n-1$. In order to prove (2), we generate the tree $\mathcal{T}_n$ inductively as follows: we start with vertex 1 by setting $\mathcal{T}_1 = \varnothing$ and $v_1 = 1$. Given $\mathcal{T}_i$, we consider the transposition $(v_{i+1}\, s_{i+1})$, where v_{i+1} is the unique minimal element contained in $\mathcal{T}_n \setminus \mathcal{T}_i$, having minimal distance to 1, and s_{i+1} is its unique $\mathcal{T}_i$-neighbor. We then set $\mathcal{T}_{i+1} = \mathcal{T}_i \cup \{(v_{i+1}\, s_{i+1})\}$. This process gives rise to the sequence of trees $\mathcal{T}_2 \subset \mathcal{T}_3 \subset \cdots \subset \mathcal{T}_n$ and denoting the vertex sets of $\mathcal{T}_i$ by V_i, we have $V_1 = \{1\} \subset V_2 \subset V_3 \subset \dots V_{n-1} \subset V_n = [n]$ where $\{v_i\} = V_i \setminus V_{i-1}$. By construction

$$\forall\, j < i; \quad x_{v_i} = ((x_1, \dots, x_n) \cdot (v_j\, s_j))_{v_i} \quad \neq \quad ((x_1, \dots, x_n) \cdot (v_i\, s_i))_{v_i},$$

where $(x_1, \dots, x_n) \cdot (v_j\, s_j)$ is the product of permutations and $((\tilde{x}_1, \dots, \tilde{x}_n))_{v_i} = \tilde{x}_{v_i}$. In other words, we order the T_n-transpositions via the sequence of trees $\{\mathcal{T}_i\}$, such that the transpositions added before $(v_i\, s_i)$ will not transpose the element x_{v_i}. To prove (3) we can, without loss of generality, restrict ourselves to the case where we have an arbitrary permutation $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$, the unique permutation satisfying $y_{v_i} = i$. We proceed by constructing a $\Gamma(S_n, T_n)$-path between these two permutations. Obviously, there exists a unique v_j such that $n = x_{v_j}$ and in the tree $\mathcal{T}_n$ there exists a unique path of length at most $\mathrm{diam}(\mathcal{T}_n) \le n-1$ connecting v_j and v_n.

Accordingly, there is a $\Gamma(S_n, T_n)$-path of length at most $\operatorname{diam}(\mathcal{T}_n)$ between (x_i) and a permutation (z_i) such that $z_{v_n} = n$. Our construction in (2) implies

$$\forall\, i < n; \quad ((z_1, \dots, z_n) \cdot (v_i\, s_i))_{v_n} = n,$$

whence we can proceed inductively, moving $(n-1)$ to the v_{n-1}th position using the subtree $\mathcal{T}_{n-1}$. We consequently arrive at

$$\operatorname{diam}(\Gamma(S_n, T_n)) \leq \sum_{i=2}^{n} \operatorname{diam}(\mathcal{T}_i) \leq \binom{n}{2}$$

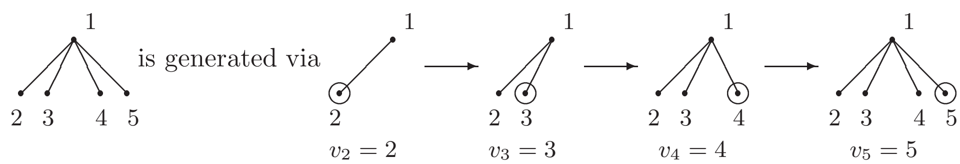
and the proof of the lemma is complete. □

In case of star-transpositions, i.e. $T_n = P_n = \{(1\, j) \mid 2 \leq j \leq n\}$, we have the following situation:

$$\{1\} \subset \{(1\,2)\} \subset \{(1\,2), (1\,3)\} \subset \cdots \subset \{(1\,j) \mid 2 \leq j \leq n\}, \tag{3.5}$$

$(v_i\, s_i) = (i\, 1)$ i.e. $s_i = 1$ and $\operatorname{diam}(\Gamma(S_n, P_n)) = \lfloor \frac{3(n-1)}{2} \rfloor$, which can be derived from a theorem of Pak [37], being strictly less than $\binom{n}{2}$.

Example 1. Consider the Cayley graph $\Gamma(S_5, P_5)$ and generate the trees $\{\mathcal{T}_i\}_{i=1}^{5}$ inductively. Setting $\mathcal{T}_1 = \varnothing$ and $v_1 = 1$ we select the minimal element in distance 1 to v_1 and set $v_2 = 2$, $\mathcal{T}_2 = \{(1\,2)\}$. We proceed by selecting the minimal element in distance 1 to the vertex set $\{1, 2\}$ and set $v_3 = 3$, $\mathcal{T}_3 = \{(1\,2), (1\,3)\}$. Finally, we select the minimal element in distance 1 to the vertex set $\{1, 2, 3\}$ and set $v_4 = 4$, $\mathcal{T}_4 = \{(1\,2), (1\,3), (1\,4)\}$. The only remaining vertex $v_5 = 5$ is the minimal element in distance 1 to the vertex set $\{1, 2, 3, 4\}$ and $\mathcal{T}_5 = \{(1\,2), (1\,3), (1\,4), (1\,5)\}$.



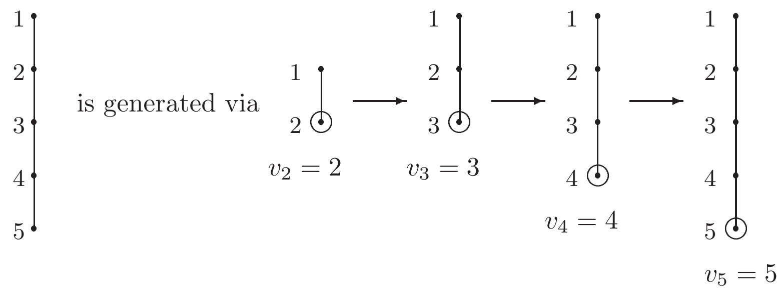
Lemma 2 provides the upper bound $\sum_{i=2}^{5} \operatorname{diam}(\mathcal{T}_i) = 7$, where $\operatorname{diam}(\Gamma(S_5, P_5)) = 6$ and the distance between $id = (1, 2, 3, 4, 5)$ and $(1, 3, 2, 5, 4)$ is the diameter of $\Gamma(S_5, P_5)$.

We next discuss the bubble-sort graph, $T_n = B_n = \{(i\, i+1) \mid 1 \leq i \leq n-1\}$. In view of

$$\{1\} \subset \{(1\,2)\} \subset \{(1\,2), (2\,3)\} \subset \cdots \subset \{(i\, i+1) \mid 1 \leq i \leq n-1\} \tag{3.6}$$

we arrive at $(v_i\, s_i) = (i\, i-1)$ and $\operatorname{diam}(\Gamma(S_n, B_n)) = \binom{n}{2}$.

Example 2. In order to make the above explicit we consider the Cayley graph $\Gamma(S_5, B_5)$ and generate the trees $\{\mathcal{T}_i\}_{i=1}^5$ inductively. Setting $\mathcal{T}_1 = \varnothing$ and $v_1 = 1$, we select the minimal element in distance 1 to v_1 and set $v_2 = 2$, $\mathcal{T}_2 = \{(1\,2)\}$. We proceed by selecting the minimal element in distance 1 to the vertex set $\{1, 2\}$ and set $v_3 = 3$, $\mathcal{T}_3 = \{(1\,2), (2\,3)\}$. Finally we select the minimal element in distance 1 to the vertex set $\{1, 2, 3\}$ and set $v_4 = 4$, $\mathcal{T}_4 = \{(1\,2), (2\,3), (3\,4)\}$. Then $v_5 = 5$ is the minimal element in distance 1 to the vertex set $\{1, 2, 3, 4\}$ and $\mathcal{T}_5 = \{(1\,2), (2\,3), (3\,4), (4\,5)\}$.



Lemma 2 provides the upper bound $\sum_{i=2}^5 \mathrm{diam}(\mathcal{T}_i) = 10$, and $\mathrm{diam}(\Gamma(S_5, B_5)) = 10$. The distance between $id = (1, 2, 3, 4, 5)$ and $(5, 4, 3, 2, 1)$ is the diameter of $\Gamma(S_5, B_5)$.

Lemma 3. *Suppose T_n is a minimal generating set of transpositions. We select permutations with independent probability $\lambda_n = \frac{1+\epsilon_n}{n-1}$, where $n^{-\frac{1}{3}+\delta} \le \epsilon_n$, for some $\delta > 0$. Then each permutation, v, is contained in a Γ_n-subtree $\mathcal{T}_n(v)$ of size $\lfloor \frac{1}{4} n^{\frac{2}{3}} \rfloor$ with probability at least $\wp(\epsilon_n)$.*

Proof. We construct the subtree $\mathcal{T}_n(v)$ by means of a branching process [25] within $\Gamma(S_n, T_n)$. Without loss of generality, we may initiate the process at id and have $r_n = n - \frac{1}{2} n^{\frac{2}{3}} \in \mathbb{N}$. We shall begin by specifying an appropriate move-set (of transpositions) by which the offspring of the branching process is being generated. To this end, let

$$N = \{(v_j\, s_j) \mid 1 \le j \le n - \frac{1}{2} n^{\frac{2}{3}} - 1\} \subset T_n.$$

Note that N acts trivially on labels v_h where $h > n - \frac{1}{2} n^{\frac{2}{3}} - 1$.
The process is defined as follows: we set $U_0 = \varnothing \subset N$ and $M_0 = L_0 = \{id\} \subset S_n$. At step $(j+1)$, suppose we are given $U_j \subset N$, M_j and $L_j \subset S_n$. In case of $L_j = \varnothing$ or $|U_j| = \lfloor \frac{1}{4} n^{\frac{2}{3}} \rfloor - 1$ the process stops. Otherwise, we consider the smallest element $l_j \in L_j$ and select among its smallest $(n - \lfloor \frac{3}{4} n^{2/3} \rfloor - 1)$ neighbors, contained in $N \setminus U_j$ with independent probability λ_n. Let $x_1 = l_j\, r_{x_1}$ be the first selected l_j-neighbor and $r_{x_1} \in N \setminus U_j$. We then set $U_j(x_1) = U_j \dot\cup \{r_{x_1}\}$ and proceed the selection with the smallest $(n - \lfloor \frac{3}{4} n^{2/3} \rfloor - 1)$ neighbors contained in $N \setminus U_j(x_1)$ instead of those

in $N \setminus U_j$. After all l_j neighbors are checked and given that $(x_1, \dots, x_s)$ have been subsequently selected, we set

$$\begin{aligned} U_{j+1} &= U_j \dot{\cup} \{r_{x_1}, \dots, r_{x_s}\} \\ L_{j+1} &= (L_j \setminus \{l_j\}) \cup \{x_1, \dots, x_s\} \\ M_{j+1} &= M_j \dot{\cup} \{x_1, \dots, x_s\}. \end{aligned}$$

The minimality of T_n and the fact that each T_n-element is used at most once implies that this process generates a tree, i.e. each M_{j+1}-element is considered only once. Furthermore, in view of

$$\frac{1+\epsilon_n}{n-1} \cdot \left(n - \lfloor \frac{3}{4} n^{\frac{2}{3}} \rfloor - 1\right) > 1. \tag{3.7}$$

Relating our construction with the binomial branching process $\mathcal{P}_m(\frac{1+\epsilon_n}{n-1})$, where $m = n - \lfloor \frac{3}{4} n^{\frac{2}{3}} \rfloor - 1$, we observe

$$\mathbb{P}\left(|M_j| = \lfloor \frac{1}{4} n^{\frac{2}{3}} \rfloor \mid \text{for some } j\right) \geq \pi_m \left(\frac{1+\epsilon_n}{n-1}\right) = \wp(\epsilon_n).$$

Indeed, the above equation holds for $\epsilon_n \geq n^{-\frac{1}{3}+\delta}$. In case of $0 < \epsilon_n = o(1)$ we notice $\lfloor \frac{3}{4} n^{\frac{2}{3}} \rfloor = o(n \cdot \epsilon_n)$. Therefore Lemma 1, (2) implies $\pi_m(\frac{1+\epsilon_n}{n-1}) = (2 + o(1))\epsilon_n = \wp(\epsilon_n)$. In case of $0 < \epsilon_n = \epsilon < 1$, we consider the probability generating functions for both: the binomial distribution, $\mathcal{P}_m(\frac{1+\epsilon}{n-1})$ and the Poisson distribution, $\mathcal{P}_P(1+\epsilon)$. Let $f_{n-1}(s)$ be the probability generating function for the binomial distribution $\text{Bi}(n-1, \frac{1+\epsilon}{n-1})$ and $g_{1+\epsilon}(s)$ be the probability generating function for Poisson distribution with parameter $\lambda = 1 + \epsilon$, i.e.

$$\begin{aligned} f_{n-1}(s) &= \sum_{j=0}^{n-1} P(\xi_i^{(t)} = j) \cdot s^j \\ &= \sum_{j=1}^{n-1} \binom{n-1}{j} \left(\frac{1+\epsilon}{n-1}\right)^j \left(1 - \frac{1+\epsilon}{n-1}\right)^{n-j} s^j \\ &= \left[1 - (1-s)\frac{1+\epsilon}{n-1}\right]^{n-1} \\ g_{1+\epsilon}(s) &= \sum_{i=0}^{\infty} e^{-(1+\epsilon)} \cdot \frac{(1+\epsilon)^i}{i!} \cdot s^i = e^{(s-1)(1+\epsilon)}. \end{aligned}$$

Clearly, $f_{n-1}(1-s) = g_{1+\epsilon}(1-s)e^{O(\frac{1}{n-1})}$ and $f_m(1-s) = f_{n-1}(1-s) \cdot (1 - s\frac{1+\epsilon}{n-1})^{-\lfloor \frac{3}{4} n^{\frac{2}{3}} \rfloor}$. By studying the roots of $f_m(1-s) = 1-s$, $f_{n-1}(1-s) = 1-s$ and $g_{1+\epsilon}(1-s) = 1-s$, we derive

$$\pi_m\left(\frac{1+\epsilon}{n-1}\right) = (1+o(1))\pi_{n-1}\left(\frac{1+\epsilon}{n-1}\right) = (1+o(1))\pi_P(1+\epsilon) = \wp(\epsilon)$$

and the lemma follows. □

For given δ, by choosing k sufficiently large, we proceed by enlarging the trees of Lemma 3 to subcomponents of arbitrary polynomial size. We remark that Lemma 2 is of central importance for the construction of the subcomponents of Lemma 4.

Lemma 4. *Given $k \geq 2$ and $\delta > 0$, $\lambda_n = \frac{1+\epsilon_n}{n-1}$, where $n^{-\frac{1}{3}+\delta} \leq \epsilon_n$, there exists a function $\theta_{n,k}$, with the property $\theta_{n,k} \geq \frac{1}{4k(k+1)} n^{\delta}$. Then each Γ_n-vertex is contained in a Γ_n-subcomponent of size at least*

$$\frac{1}{2^{k+2}} \cdot \left[\frac{1}{4k(k+1)}\right]^k \cdot n^{\frac{2}{3}+k\delta}$$

with probability at least

$$\delta_k(\epsilon_n) = \wp(\epsilon_n)\,(1 - e^{-\beta_{k,n}\theta_{n,k}}), \tag{3.8}$$

where $0 < \beta_{k,n} < 1$ and $\epsilon_n \geq n^{-\frac{1}{3}+\delta}$.

Proof. Without loss of generality we may assume $\pi = id$, $\mu_n \in \mathbb{N}$ and set for all $1 \leq m \leq k$,

$$A_m = \left\{ (v_j^m\, s_j^m) \in T_n \mid 1 \leq j \leq \mu_n \right\}.$$

where $(v_j^m\, s_j^m) = (v_{r_n+j+(m-1)\mu_n-1}\, s_{r_n+j+(m-1)\mu_n-1})$ and $r_n = n - \lfloor \frac{1}{2} n^{\frac{2}{3}} \rfloor$, see eq. (3.2). That is, A_m is the "first" (in the sense of the labeling given by the sequence $(v_{r_n}, v_{r_n+1}, \ldots, v_n)$) subset of T_n-transpositions that act on labels v_i, where $i \leq r_n + m\mu_n - 1$ for $1 \leq m \leq k$. Furthermore, for $1 \leq m \leq k$, $|A_m| = \mu_n = \lfloor \frac{1}{2k(k+1)} n^{\frac{2}{3}} \rfloor$, see eq. (3.2). We set $w_j^{(h)} = (v_j^h\, s_j^h) \in A_h$ and consider the branching process of Lemma 3 at $\pi = id$, assuming that we obtain a tree T^1 of size $\lfloor \frac{1}{4} n^{\frac{2}{3}} \rfloor$. Let

$$Y_1 = \left| \{ w_i^{(1)} \in A_1 \mid \exists x \in T^1; x \cdot w_i^{(1)} \in \Gamma_n \} \right|.$$

According to Lemma 2

$$\forall\, x, y \in T^1; \forall\, w_i^{(1)} \neq w_r^{(1)} \in A_1; \quad x \cdot w_i^{(1)} \neq y \cdot w_r^{(1)},$$

whence

$$\mathbb{E}[Y_1] = \mu_n \cdot \left(1 - \left(1 - \frac{1+\epsilon_n}{n-1} \right)^{\frac{1}{4} n^{\frac{2}{3}}} \right) \sim \mu_n \left(1 - \exp(-(1+\epsilon_n)\frac{1}{4} n^{-\frac{1}{3}}) \right). \tag{3.9}$$

Using large deviation inequalities eq. (2.4) [14], we conclude that $\beta_1 = \frac{1}{8} > 0$ satisfies

$$\mathbb{P}\left(Y_1 < \frac{1}{2}\mathbb{E}[Y_1] \right) \leq \exp\left(-\beta_1 \cdot \mathbb{E}[Y_1] \right).$$

We select the smallest element, $x_{(i\,j)}$, from the set $\{x \cdot w_j^{(1)} \mid x \in T^1, x \cdot w_j^{(1)} \in \Gamma_n\}$ and start the branching process of Lemma 3 at $x_{(i\,j)}$. As a result, we derive the tree $C_2(x_{(i\,j)})$ of size $\lfloor \frac{1}{4} n^{\frac{2}{3}} \rfloor$ with probability at least $\wp(\epsilon_n)$. However, note that $T^1 \cup C_2(x_{(i\,j)})$ may not be tree any more. According to Lemma 3, the generation of this tree $C_2(x_{(i\,j)})$ exclusively involves labels v_j where $j \le r_n - 1$. Therefore, since any two smallest elements $x_{(i_1\,j_1)}$ and $x_{(i_2\,j_2)}$ differ in at least one of two coordinates with labels v_{j_1}, v_{j_2} for $r_n \le j_1, j_2 \le r_n + \mu_n$, we have

$$C_2(x_{(i_1\,j_1)}) \cap C_2(x_{(i_2\,j_2)}) = \varnothing.$$

Let X_1 be the r.v. counting the number of these new Γ_n-subcomponents. In view of eq. (3.9), we obtain

$$\mathbb{E}[X_1] = \wp(\epsilon_n) \cdot \mathbb{E}[Y_1] \sim \wp(\epsilon_n) \cdot \mu_n \left(1 - \exp(-(1+\epsilon_n)\frac{1}{4} n^{-\frac{1}{3}})\right).$$

In order to make the dependence of $\theta_{n,k} = \wp(\epsilon_n) \cdot \mu_n \left(1 - \exp(-(1+\epsilon_n)\frac{1}{4} n^{-\frac{1}{3}})\right)$ for fixed $\delta > 0$ on k and n explicit, we compute

$$\begin{aligned} \theta_{n,k} &\ge 2 \cdot n^{-\frac{1}{3}+\delta} \cdot \frac{1}{2k(k+1)} n^{\frac{2}{3}} \cdot (1 + n^{-\frac{1}{3}+\delta}) \cdot \frac{1}{4} \cdot n^{-\frac{1}{3}} - o(1) \\ &= \frac{1}{4k(k+1)} \cdot n^{\delta} \qquad \text{as } n \to \infty. \end{aligned}$$

Again, using large deviation inequalities eq. (2.4), we conclude that $\beta_1 = \frac{1}{8} > 0$ satisfies

$$\mathbb{P}(X_1 < \frac{1}{2}\theta_{n,k}) \le \exp(-\beta_1 \theta_{n,k})$$

or equivalently, since the union of all the $C_2(x_{(i\,j)})$-subcomponents with T^1 forms a $\Gamma(S_n, T_n)$-subcomponent, T^2, we have

$$\mathbb{P}\left(|T^2| < \lfloor \frac{1}{4} n^{2/3} \rfloor \cdot \frac{1}{2}\theta_{n,k}\right) \le \exp(-\beta_1 \theta_{n,k}). \tag{3.10}$$

We now proceed by induction:
Claim: For each $2 \le i \le k$, there exists some constant $\beta_{i,n} > 0$ and a $\Gamma(S_n, T_n)$-subcomponent T^i such that

$$\mathbb{P}(|T^i| < \lfloor \frac{1}{4} n^{2/3} \rfloor \cdot \left(\frac{\theta_{n,k}}{2}\right)^{i-1}) \le \exp(-\beta_{i-1,n} \theta_{n,k}).$$

We have already established the induction basis. As for the induction step, let us assume the claim holds for $i < k$ and let $C_i(\alpha)$ denote a subcomponent generated by the branching process of

Lemma 3 in the i-th step. We consider the T_n-transpositions $w_r^{(i+1)} \neq w_a^{(i+1)} \in A_{i+1}$. We consider the minimal elements, x_r^α of

$$Y_{i+1} = \{w_r^{(i+1)} \in A_{i+1} \mid \exists\, x \in C_i(\alpha); x \cdot w_r^{(i+1)} \in \Gamma_n\}$$

at which we initiate the branching process of Lemma 3. The process generates subcomponents $C_{i+1}(x_r^\alpha)$ of size $\lfloor \frac{1}{4} n^{\frac{2}{3}} \rfloor$ with probability $\geq \wp(\epsilon_n)$. Any two of these are mutually disjoint and let X_{i+1} be the r.v. counting their number. We derive setting $q_n = \lfloor \frac{1}{4} n^{2/3} \rfloor$. In order to make the dependence of $\beta_{i,n}$ for fixed $\delta > 0, k \geq 2$ on n and i explicit, we set $\beta_{1,n} = \beta_1 = \frac{1}{8}$ and recursively define $\beta_{i,n}$ for $i \geq 2$,

$$\beta_{i,n} = \beta_{i-1,n} - \frac{\ln(1 + \exp(-\beta_1 \theta_{n,k}^{i-1} + \beta_{i-1,n}\theta_{n,k}))}{\theta_{n,k}} = \beta_{i-1,n} + o(1) \quad \text{for } k \geq i \geq 2$$

We compute

$$\begin{aligned}
\mathbb{P}\left(|T^{i+1}| < q_n \frac{1}{2^i} \theta_{n,k}^i\right) \quad &\leq \quad \underbrace{\mathbb{P}\left(|T^i| < q_n \frac{1}{2^{i-1}} \theta_{n,k}^{i-1}\right)}_{\text{failure at step } i} + \\
&\quad\;\; \underbrace{\mathbb{P}\left(|T^{i+1}| < q_n \frac{1}{2^i} \theta_{n,k}^i \text{ and } |T^i| \geq q_n \frac{1}{2^{i-1}} \theta_{n,k}^{i-1}\right)}_{\text{failure at step } i+1 \text{ conditional to } |T^i| \geq q_n \frac{1}{2^{i-1}} \theta_{n,k}^{i-1}} \\
&\leq \quad \underbrace{e^{-\beta_{i-1,n}\,\theta_{n,k}}}_{\text{induction hypothesis}} + \underbrace{e^{-\beta_1\,\theta_{n,k}^i}}_{\text{large deviation results}} \cdot (1 - e^{-\beta_{i-1,n}\,\theta_{n,k}}), \\
&\leq \quad e^{-\beta_{i,n}\,\theta_{n,k}}
\end{aligned}$$

and the Claim follows.
Therefore, each Γ_n-vertex is contained in a subcomponent of size

$$\geq \frac{1}{4} \cdot n^{\frac{2}{3}} \cdot \frac{1}{2^k} \cdot \left[\frac{1}{4k(k+1)}\right]^k \cdot n^{k\delta} = \frac{1}{2^{k+2}} \cdot \left[\frac{1}{4k(k+1)}\right]^k \cdot n^{\frac{2}{3}+k\delta},$$

with probability at least $\wp(\epsilon_n)(1 - e^{-\beta_{k,n}\theta_{n,k}})$ and the lemma is proved. □

4. Vertices in small components

For given $0 < \delta < 1$, let

$$M_k(n) = \frac{1}{2^{k+2}} \left[\frac{1}{4k(k+1)}\right]^k n^{\frac{2}{3}+k\delta}. \tag{4.1}$$

Let $\Gamma_{n,k}$ denote the set of Γ_n-vertices contained in components of size $\geq M_k(n)$ for fixed $0 < \delta < 1$. In this section we prove that $|\Gamma_{n,k}|$ is a.s. $\sim \wp(\epsilon_n)\frac{1+\epsilon_n}{n-1}n!$. In analogy to Lemma 3 of [38] we first observe that the number of vertices, contained in Γ_n-components of size $< M_k(n)$, is sharply concentrated. The concentration reduces the problem to a computation of expectation values. It follows from considering the indicator r.vs. of pairs (C, v) where C is a component and $v \in C$ and to estimate their correlation. Since the components in question are small, no "critical" correlation terms arise.

Let $U_n = U_n(a)$ denote the set of vertices contained in components of size $< n^a$ where $a > 0$. Then following the arguments in [10]

Lemma 5. *Let $a > 0$ be a fixed constant. We are given $\delta > 0$ and $\lambda_n = \frac{1+\epsilon_n}{n-1}$, where $1 > \epsilon_n \geq n^{-\frac{1}{3}+\delta}$. Then*

$$\mathbb{P}\left(\,\big|\,|U_n| - \mathbb{E}[|U_n|]\,\big| \geq \frac{1}{n}\mathbb{E}[|U_n|]\right) = o(1). \tag{4.2}$$

Proof. Let $I_{C,v}$, be the indicator r.v. of the pair (C, v), where $v \in C$ and $C \in U_n$ is a component of size $< n^a$. We have

$$|U_n| = \sum_{(C,v)} I_{C,v}.$$

and we proceed by proving that the r.v. $|U_n|$ is sharply concentrated by analyzing the correlation terms $\mathbb{E}(I_{C_1,v}I_{C_2,w})$. Correlation may arise in two ways: the pairs (C_1, v) and (C_2, w) either satisfy $C_1 = C_2$ or the minimal distance, $d_{\Gamma(S_n,T_n)}(C_1, C_2) = 2$. Suppose first $C_1 = C_2$, then

$$\begin{aligned}
\sum_{(C,v)\sim(C,w)} \mathbb{E}(I_{C,v}I_{C,w}) &= \sum_{(C,v)} \sum_{(C,w)\sim(C,v)} \mathbb{E}(I_{C,v}) \\
&\leq \sum_{(C,v)} n^a\mathbb{E}(I_{C,v}) = n^a\mathbb{E}[|U_n|]
\end{aligned}$$

Secondly we consider the case $C_1 \neq C_2$. Then there exist vertices $v \in C_1$ and $w \in C_2$ with $d_{\Gamma(S_n,T_n)}(v, w) = 2$, i.e. we have an additional vertex $u \notin \Gamma_n$ which, if selected, would lead to a merger of the subcomponents C_1 and C_2. Accordingly,

$$\begin{aligned}
\mathbb{P}(d(C_1, C_2) = 2) &= \frac{(1 - \lambda_n)}{\lambda_n}\,\mathbb{P}(C_1 \cup C_2 \cup \{u\} \text{ is a } \Gamma_n\text{-component}) \\
&\leq n\,\mathbb{P}(C_1 \cup C_2 \cup \{u\} \text{ is a } \Gamma_n\text{-component})
\end{aligned}$$

and we derive, summing over all possible v, w, u, the upper bound

$$\sum_{d(C_1,C_2)=2} \mathbb{E}[I_{C_1,v_1}\, I_{C_2,v_2}] \le n\,(2n^a+1)^3\,|\Gamma_n|.$$

The uncorrelated pairs $(I_{C_1,v_1}, I_{C_2,v_2})$ can be estimated by

$$\sum_{(C_1,v_1)\not\sim(C_2,v_2)} \mathbb{E}[I_{C_1,v_1}\, I_{C_2,v_2}] = \sum_{(C_1,v_1)\not\sim(C_2,v_2)} \mathbb{E}[I_{C_1,v_1}]\cdot\mathbb{E}[I_{C_2,v_2}] \le \mathbb{E}[|U_n|]^2.$$

Consequently we arrive at

$$\begin{aligned}\mathbb{E}[U_n(U_n-1)] &= \sum_{\substack{(C,v_1)\\ \sim(C,v_2)}} \mathbb{E}[I_{C,v_1}\, I_{C,v_2}] + \sum_{\substack{(C_1,v_1)\\ \sim(C_2,v_2)}} \mathbb{E}[I_{C_1,v_1}\, I_{C_2,v_2}] + \sum_{\substack{(C_1,v_1)\\ \not\sim(C_2,v_2)}} \mathbb{E}[I_{C_1,v_1}\, I_{C_2,v_2}] \\ &\le n^a\,\mathbb{E}[|U_n|] + n\,(2n^a+1)^3|\Gamma_n| + \mathbb{E}[|U_n|]^2.\end{aligned}$$

Just considering isolated vertices implies $\mathbb{E}[U_n] \ge c\,|\Gamma_n|$ for some $c > 0$, i.e. the expected number of vertices in small components grows faster than any polynomial. Employing Chebyshev's inequality, eq. (2.5), we derive

$$\begin{aligned}\mathbb{P}\left(||U_n| - \mathbb{E}[|U_n|]| \ge \frac{1}{n}\,\mathbb{E}[|U_n|]\right) &\le n^2\,\frac{\mathbb{V}[|U_n|]}{\mathbb{E}[|U_n|]^2} \\ &= n^2\,\frac{\mathbb{E}[|U_n|(|U_n|-1)] + \mathbb{E}[|U_n|] - \mathbb{E}[|U_n|]^2}{\mathbb{E}[|U_n|]^2} \\ &\le n^2\,\frac{n^a + \frac{1}{c}\,n\,(2n^a+1)^3+1}{\mathbb{E}[|U_n|]} = o\left(\frac{1}{n^2}\right),\end{aligned}$$

whence the lemma. □

With the help of Lemma 5, we proceed by computing the size of $\Gamma_{n,k}$.

Lemma 6. *Suppose $k \in \mathbb{N}$ is arbitrary but fixed and we are given $\delta > 0$. Let $\omega_n = |\Gamma_n \backslash \Gamma_{n,k}|$ and $\lambda_n = \frac{1+\epsilon_n}{n-1}$, where $n^{-\frac{1}{3}+\delta} \le \epsilon_n < 1$. Then*

$$|\Gamma_{n,k}| \sim \wp(\epsilon_n)\frac{1+\epsilon_n}{n-1}n! \qquad a.s.\ . \tag{4.3}$$

Proof. First we prove for any $n^{-\frac{1}{3}+\delta} \le \epsilon_n \le \lambda$, where $\lambda > 0$

$$(1-o(1))\wp(\epsilon_n)\,|\Gamma_n| \le |\Gamma_{n,k}| \qquad \text{a.s.} \tag{4.4}$$

By Lemma 4 we have

$$\mathbb{E}[\omega_n] \le (1-\delta_k(\epsilon_n))|\Gamma_n|.$$

In view of Lemma 5, we derive

$$\omega_n < \left(1 + O(\frac{1}{n})\right) \mathbb{E}[\omega_n] < \left(1 - \delta_k(\epsilon_n) + O(\frac{1}{n})\right) |\Gamma_n| \quad \text{a.s.,}$$

whence

$$|\Gamma_{n,k}| \geq \left(\delta_k(\epsilon_n) - O(\frac{1}{n})\right) |\Gamma_n| = (1 - o(1))\wp(\epsilon_n)|\Gamma_n| \quad \text{a.s..}$$

Next we prove for $n^{-\frac{1}{3}+\delta} \leq \epsilon_n < 1$ and arbitrary but fixed k,

$$|\Gamma_{n,k}| \leq (1 + o(1))\wp(\epsilon_n)\,|\Gamma_n| \qquad \text{a.s.} \tag{4.5}$$

Let $W_n = U_n(\frac{1}{2}) = \{r \in \Gamma(S_n, T_n) \mid |C_r| < n^{1/2}\}$, where C_r denotes a component containing r. Obviously, $\Gamma_{n,k} \subset \Gamma_n \setminus W_n$, whence it suffices to prove

$$|W_n| \geq [1 - (1 + o(1))\wp(\epsilon_n)]\ |\Gamma_n| \qquad \text{a.s.} \tag{4.6}$$

For this purpose we follow [9] and consider a certain branching process in the $(n-1)$-regular rooted tree T_{r^*}. Here the r.v. ξ_r^* of the rooted vertex r^* is $\text{Bi}(n-1, \lambda_n)$ distributed while the r.v. of any other vertex r has the distribution $\text{Bi}(n-2, \lambda_n)$. Let C_{r^*} denote the component generated by this branching process. The idea here is to relate C_{r^*} with its image under a covering map, i.e. a specific Γ_n-component containing r, denoted by C_r.

Using the linear ordering on $\Gamma(S_n, T_n)$, one can specify a unique procedure on how to generate an acyclic connected $\Gamma(S_n, T_n)$-subgraph of size $< n^{1/2}$, denoted by $H_r^\dagger$ [9]. Let S be a stack. We initialize by setting $H_r^\dagger = \{r\}$. Then we select the r-neighbors in $\Gamma(S_n, T_n)$, one by one, in increasing order, with probability λ_n. For each selected neighbor r_i, we (a) put the corresponding edge $\{r, r_i\}$ into S, (b) add r_i to $H_r^\dagger$ and (c) check condition (h1) "$|H_r^\dagger| = n^{\frac{1}{2}}$". If (h1) holds we stop, otherwise we proceed examining the next r-neighbor. Suppose (h1) does not hold and all r-neighbors have been examined.

If S is empty, we stop. Otherwise we proceed inductively as follows: we remove the first element, $\{r, w\}$ from S and consider the w-neighbors, except r, one by one, in increasing order. For each selected w-neighbor, x, we (a) insert the edge $\{w, x\}$ into the back of S (b) add x to $H_r^\dagger$ and (c) check condition (h1) "$|H_r^\dagger| = n^{\frac{1}{2}}$"and (h2) "$H_r^\dagger$ contains a cycle". In case (h1) or (h2) holds we stop. Otherwise, we continue examining w-neighbors in increasing order until all w-neighbors are considered. If S is empty we stop and otherwise we consider the next element from S and iterate the process.

Consequently we have by construction

$$\forall m \leq n^{\frac{1}{2}}; \quad \mathbb{P}\left(|H_r^\dagger| < m \text{ and } H_r^\dagger \text{ is a acyclic}\right) \leq \mathbb{P}\left(|C_{r^*}| < m\right), \tag{4.7}$$

where the discrepancy between $\mathbb{P}\left(|H_r^\dagger| < m \text{ and } H_r^\dagger \text{ is a acyclic}\right)$ and $\mathbb{P}\left(|C_{r^*}| < m\right)$ lies in those events for which a $\le$-compatible covering map from T_{r^*} into $\Gamma(S_n, T_n)$, mapping r^* into r, produces a cycle in $\Gamma(S_n, T_n)$. The latter is bounded from above by the probability $\mathbb{P}\left(H_r^\dagger \text{ contains a cycle}\right)$. Therefore,

$$\forall m \le n^{\frac{1}{2}}; \quad \mathbb{P}\left(|H_r^\dagger| < m \text{ and } H_r^\dagger \text{ is a acyclic}\right) \ge \mathbb{P}\left(|C_{r^*}| < m\right) - \mathbb{P}\left(H_r^\dagger \text{ contains a cycle}\right). \tag{4.8}$$

We proceed by computing $\mathbb{P}\left(|C_{r^*}| < m\right)$ and $\mathbb{P}(H_r^\dagger$ contains a cycle).
Claim 1:[9] there exists some $\kappa > 0$ such that

$$\mathbb{P}(|C_{r^*}| < n^{1/2}) \quad \ge \quad 1 - \pi_0(\epsilon_n) - o(e^{-\kappa\, n^{1/2}}). \tag{4.9}$$

To prove the claim we compute

$$\begin{aligned}
\mathbb{P}(n^{1/2} \le |C_{r^*}| < \infty) \quad &= \quad \sum_{i \ge n^{1/2}} \mathbb{P}(|C_{r^*}| = i) \\
&= \quad \sum_{i \ge n^{1/2}} (1 + o(1)) \cdot \frac{(\lambda_n \cdot (n-2))^{i-1}}{i\sqrt{2\pi i}} \left[\frac{(n-2)(1-\lambda_n)}{(n-3)}\right]^{ni-3i+2} \\
&\le \quad \sum_{i \ge n^{1/2}} \left[(1+\epsilon_n)e^{-\epsilon_n}\right]^i \le \sum_{i \ge n^{1/2}} c(\epsilon)^i = o(e^{-\kappa n^{1/2}}),
\end{aligned}$$

where $0 < c(\epsilon) < 1$ and

$$\mathbb{P}(|C_{r^*}| = i) = (1 + o(1)) \cdot \frac{(\lambda_n \cdot (n-2))^{i-1}}{i\sqrt{2\pi i}} \left[\frac{(n-2)(1-\lambda_n)}{(n-3)}\right]^{ni-3i+2}, \tag{4.10}$$

where $i = i(n) \to \infty$ as $n \to \infty$ is due to [9]. We accordingly derive

$$\begin{aligned}
\mathbb{P}(|C_{r^*}| < n^{1/2}) \quad &= \quad \mathbb{P}(|C_{r^*}| < \infty) - \mathbb{P}(n^{1/2} \le |C_{r^*}| < \infty) \\
&\ge \quad 1 - \underbrace{\wp(\epsilon_n)}_{=\pi_0(\frac{1+\epsilon_n}{n-1})} - o(e^{-\kappa n^{1/2}}),
\end{aligned} \tag{4.11}$$

where $\pi_0(\frac{1+\epsilon_n}{n-1}) = \wp(\epsilon_n) = \mathbb{P}(|C_{r^*}| = \infty)$ is the survival probability of the branching process in T_{r^*}, which constructs the component rooted in r^*, see Lemma 1.
Claim 2: $\mathbb{P}(H_r^\dagger$ contains a cycle$) \le O(n^{-\frac{1}{2}})$.
Let ℓ denote the length of a cycle, $\mathcal{O}_\ell$, generated by $H_r^\dagger$. We first notice that $\mathcal{O}_\ell$ contains at most $\lfloor \frac{\ell}{2} \rfloor$ distinct T_n-elements. Otherwise $\mathcal{O}_\ell = (\sigma_s)_{s=1}^{\ell}$ contains $\lfloor \frac{\ell}{2} \rfloor + 1$ distinct T_n-transpositions and consequently there exists at least one transposition $\sigma_t = (i\,j) \in \mathcal{O}_\ell$ that occurs only once. Then we conclude, using $\prod_{s=1}^{\ell} \sigma_s = 1$,

$$(i\,j) \in \langle T_n \setminus \{(i\,j)\} \rangle,$$

which is impossible since T_n is a minimal generating set. Let N be the number of distinct transpositions in $\mathcal{O}_\ell$ and a_s be the multiplicity of s-th distinct transposition. We then have $a_s \geq 2$ for $1 \leq s \leq N$ and $N \leq \lfloor \frac{\ell}{2} \rfloor$. We notice that the number of such cycles $\mathcal{O}_\ell$, that contain a fixed vertex is bounded from above by

$$\begin{aligned} & \binom{n-1}{N} \cdot \frac{\ell!}{a_1! \cdot a_2! \cdots a_N!} \\ \leq \quad & \binom{n-1}{N} \cdot \frac{\ell!}{2^N} \leq \left(\frac{n-1}{2}\right)^N \frac{\ell!}{N!} \leq \left(\frac{n-1}{2}\right)^{\lfloor \frac{\ell}{2} \rfloor} \frac{\ell!}{(\lfloor \frac{\ell}{2} \rfloor)!} = O\left(\frac{\ell(n-1)}{e}\right)^{\lfloor \frac{\ell}{2} \rfloor} \end{aligned}$$

We next distinguish the cases of whether or not $\mathcal{O}_\ell$ contains r. Let us first assume $r \notin \mathcal{O}_\ell$. Then all vertices except of the lastly added vertex w, have been examined only once while w has been examined for at most $n^{\frac{1}{2}} - 1$ times. Therefore the probability of $\mathcal{O}_\ell$ is bounded by

$$\leq n^{\frac{1}{2}} \cdot \ell \cdot \left(\frac{n-1}{2}\right)^{\lfloor \frac{\ell}{2} \rfloor} \frac{\ell!}{(\lfloor \frac{\ell}{2} \rfloor)!} \cdot \left(\frac{2}{n-1}\right)^{\ell-1} \frac{2}{n-1} \cdot \left(n^{\frac{1}{2}} - 1\right) = O\left(\ell n \cdot \left(\frac{4\ell}{e(n-1)}\right)^{\lfloor \frac{\ell}{2} \rfloor}\right).$$

Taking the sum over all possible values $4 \leq \ell \leq n^{\frac{1}{2}}$, we observe that the probability of the event that $H_r^\dagger$ contains such a cycle, is at most $O(n^{-1})$.
Suppose next $r \in \mathcal{O}_\ell$. Then r has by construction never been examined. The lastly added vertex (the one leading to the cycle and therefore to the halting of the process) has been examined at most $n^{\frac{1}{2}} - 1$ times and all other vertices contained in $\mathcal{O}_\ell$ have been examined only once. Therefore the probability of $\mathcal{O}_\ell$ is bounded by

$$\leq \ell \cdot \left(\frac{n-1}{2}\right)^{\lfloor \frac{\ell}{2} \rfloor} \frac{\ell!}{(\lfloor \frac{\ell}{2} \rfloor)!} \cdot \left(\frac{2}{n-1}\right)^{\ell-2} \frac{2}{n-1} \cdot \left(n^{\frac{1}{2}} - 1\right) = O\left(\ell n^{\frac{3}{2}} \cdot \left(\frac{4\ell}{e(n-1)}\right)^{\lfloor \frac{\ell}{2} \rfloor}\right).$$

Taking the sum over $4 \leq \ell \leq n^{\frac{1}{2}}$, we conclude that the probability of the event that $H_r^\dagger$ contains a cycle that contains r, is at most $O(n^{-\frac{1}{2}})$ and Claim 2 follows.
Claim 3*:*

$$\mathbb{P}\left(|C_r| < n^{\frac{1}{2}}\right) \geq 1 - (1 + o(1))\wp(\epsilon_n). \tag{4.12}$$

Let D_r be a tree containing r of size $< n^{\frac{1}{2}}$ in Γ_n. Since there is only one way by which the procedure $H_r^\dagger$ can generate D_r we have

$$\mathbb{P}\left(C_r = D_r\right) \geq \mathbb{P}\left(H_r^\dagger = D_r\right). \tag{4.13}$$

Consequently, taking the sum over all such trees we obtain

$$\mathbb{P}\left(|C_r| < n^{\frac{1}{2}} \text{ and } C_r \text{ is a tree}\right) \geq \mathbb{P}\left(|H_r^\dagger| < n^{\frac{1}{2}} \text{ and } H_r^\dagger \text{ is acyclic}\right). \tag{4.14}$$

According to eq. (4.8), Claim 1, Claim 2 and $\wp(\epsilon_n) \geq n^{-1/3+\delta}$ we conclude

$$\mathbb{P}\left(|H_r^\dagger| < n^{\frac{1}{2}} \text{ and } H_r^\dagger \text{ is acyclic }\right) \geq 1 - (1+o(1))\wp(\epsilon_n).$$

Accordingly we arrive at

$$\begin{aligned}
\mathbb{P}\left(|C_r| < n^{\frac{1}{2}}\right) &\geq \mathbb{P}\left(|C_r| < n^{\frac{1}{2}} \text{ and } C_r \text{ is a tree}\right) \\
&\geq \mathbb{P}\left(|H_r^\dagger| < n^{\frac{1}{2}} \text{ and } H_r^\dagger \text{ is acyclic}\right) \\
&\geq 1 - \wp(\epsilon_n) - o(e^{-\kappa n^{\frac{1}{2}}}) - O(n^{-\frac{1}{2}}) \\
&\geq 1 - (1+o(1))\wp(\epsilon_n)
\end{aligned}$$

and Claim 3 is proved. By linearity of expectation, we have $(1-(1+o(1))\wp(\epsilon_n))|\Gamma_n| \leq \mathbb{E}[|W_n|]$ and according to Lemma 5, $(1-O(n^{-1}))\,\mathbb{E}[|W_n|] < |W_n|$ a.s.. In view of $n^{-1} = o(\wp(\epsilon_n))$ we have therefore proved eq. (4.6)

$$(1-(1+o(1))\,\wp(\epsilon_n))\,|\Gamma_n| \leq |W_n| \qquad \text{a.s.}$$

and the proof of lemma is complete. □

5. The main theorem

We show in this section that the unique giant component forms within $\Gamma_{n,k}$ for two reasons: first, for given δ, any $\Gamma_{n,k}$-vertex is *a priori* contained in a subcomponent of size $\geq M_k(n)$, see eq. (4.1), limiting the number of ways by which $\Gamma_{n,k}$-splits can be chosen and second there are many independent paths connecting large $\Gamma(S_n, T_n)$-subsets. We first prove Lemma 7 according to which $\Gamma_{n,k}$ is "almost" 2-dense in $\Gamma(S_n, T_n)$.

Lemma 7. *Let $k \in \mathbb{N}$ and $\Delta_k = \left[\frac{k}{2(k+1)}\right]^2/2$, $\lambda_n = \frac{1+\epsilon_n}{n-1}$ where $\epsilon_n \geq n^{-\frac{1}{3}+\delta}$ for some $\delta > 0$ and let furthermore $A_\delta = \left\{v \mid |d(v,2) \cap \Gamma_{n,k}| < \frac{1}{2}\Delta_k \cdot n^\delta\right\}$. Then $\mathbb{P}(v \in A_\delta) \leq \exp(-\frac{1}{8}\Delta_k \cdot n^\delta)$ and there exists some $0 < \rho_k < \frac{1}{8}\Delta_k$ for arbitrary but fixed k, such that*

$$|A_\delta| \leq n! e^{-\rho_k n^\delta} \quad \textit{a.s.}.$$

Proof. We consider now the action of the transpositions

$$A_{k+1} = \left\{(v_j^{k+1}\, s_j^{k+1}) \in T_n \mid 1 \leq j \leq \ell_n\right\}$$

where $w_j^{(k+1)} = (v_j^{k+1}\, s_j^{k+1}) = (v_{r_n-1+j+k\mu_n}\, s_{r_n-1+j+k\mu_n})$ and $\ell_n = \lfloor \frac{k}{2(k+1)} n^{\frac{2}{3}} \rfloor$, see eq. (3.2) and set

$$d^{(k+1)}(v,2) = \{v \cdot w_i^{(k+1)} \cdot w_j^{(k+1)} | 1 \le i < j \le \ell_n\}.$$

We proceed by establishing a lower bound on the cardinality of $d^{(k+1)}(v,2)$. Since T_n is a minimal generating set, any sequence of distinct T_n-transpositions is acyclic. Therefore

$$|d^{(k+1)}(v,2)| \quad \ge \quad \binom{\ell_n}{2} = \frac{n^{\frac{4}{3}}}{2} \cdot \left[\frac{k}{2(k+1)}\right]^2 \cdot (1 - o(1)).$$

Let $\Delta_k = \left[\frac{k}{2(k+1)}\right]^2 /2$ and $Z(v)$ be the r.v. counting the number of vertices contained in the set $d^{(k+1)}(v,2) \cap \Gamma_{n,k}$, whose subcomponents are constructed in Lemma 4. We immediately compute

$$\mathbb{E}(Z(v)) \ge \lambda_n \cdot \delta_k(\epsilon_n) \cdot |d^{(k+1)}(v,2)| \sim \Delta_k\, n^{\frac{4}{3}} \cdot \frac{1+\epsilon_n}{n-1} \cdot \wp(\epsilon_n)(1 - e^{-\beta_{k,n}\theta_{n,k}}) \ge \Delta_k \cdot n^{\delta}.$$

The key observation is the following: the construction of the Lemma 4-subcomponents did not involve any labels $v_{r_n-1+j+k\mu_n}$, i.e. any two such subcomponents remain vertex-disjoint. Therefore the r.v. $Z(v)$ is a sum of *independent* indicator r.vs. and Chernoff's large deviation inequality, eq. (2.4), [14] implies

$$\mathbb{P}(v \in A_\delta) = \mathbb{P}\left(Z(v) < \frac{1}{2}\,\Delta_k \cdot n^{\delta}\right) \le \exp(-\frac{1}{8}\Delta_k \cdot n^{\delta}). \tag{5.1}$$

Consequently, the expected number of vertices contained in A_δ is bounded by $n! \exp(-\frac{1}{8}\Delta_k \cdot n^{\delta})$. Now Markov's inequality [39],

$$\mathbb{P}(X > t\mathbb{E}(X)) \le 1/t, \quad t > 0,$$

guarantees $|A_\delta| \le n! \cdot e^{-\rho_k n^{\delta}}$ a.s. for any $0 < \rho_k < \frac{1}{8}\Delta_k$ and arbitrary, fixed k and the lemma follows. □

Next we show that there exist many vertex disjoint paths between $\Gamma_{n,k}$-splits of sufficiently large size. The proof is analogous to Lemma 7 in [38]. We remark that Lemma 8 does not use an isoperimetric inequality [24]. It only employs a generic estimate of the vertex boundary in Cayley graphs due to Aldous [4, 6].

Lemma 8. *Let (S,T) be a vertex-split of $\Gamma_{n,k}$ with the properties*

$$\exists\, 0 < \rho_0 \le \rho_1 < 1; \quad (n-2)! \le |S| = \rho_0 |\Gamma_{n,k}| \quad \text{and} \quad (n-2)! \le |T| = \rho_1 |\Gamma_{n,k}|. \tag{5.2}$$

Then there exists some $c > 0$ such that a.s. $d(S)$ is connected to $d(T)$ in $\Gamma(S_n, T_n)$ via at least

$$c\,(n-5)!/(n-1)^7 \tag{5.3}$$

vertex disjoint (independent) paths of length ≤ 3.

Proof. We distinguish the cases $|\mathsf{B}(S,2)| \leq \frac{2}{3}\, n!$ and $|\mathsf{B}(S,2)| > \frac{2}{3}\, n!$. In the former case, we employ the generic estimate of vertex boundaries in Cayley graphs [4]

$$|\mathsf{d}(S)| \geq \frac{1}{\mathrm{diam}(\Gamma(S_n,T_n))} \cdot |S| \left(1 - \frac{|S|}{n!}\right). \tag{5.4}$$

In view of eq. (5.2) and Lemma 2, eq. (5.4) implies

$$\exists\, d_1 > 0; \quad |\mathsf{d}(\mathsf{B}(S,2))| \geq \frac{d_1}{n^2} \cdot |\mathsf{B}(S,2)| \geq d_1 \cdot (n-4)!. \tag{5.5}$$

According to Lemma 7, a.s. all but $\leq n!\, e^{-\rho_k n^\delta}$ permutations are within distance 2 to some $\Gamma_{n,k}$-vertex, whence

$$|\mathsf{d}(\mathsf{B}(S,2)) \cap \mathsf{B}(T,2)| \geq d_2 \cdot (n-4)! \quad \text{a.s..} \tag{5.6}$$

Let $\beta_2 \in \mathsf{d}(\mathsf{B}(S,2)) \cap \mathsf{B}(T,2)$. Then there exists a path $(\alpha_1, \alpha_2, \beta_2)$ such that $\alpha_1 \in \mathsf{d}(S)$, $\alpha_2 \in \mathsf{d}(B(S,1))$. We distinguish the cases

$$|\mathsf{d}(\mathsf{B}(S,2)) \cap \mathsf{d}(\mathsf{B}(T,1))| \geq d_{2,1}\,(n-4)! \quad \text{and} \quad |\mathsf{d}(\mathsf{B}(S,2)) \cap \mathsf{B}(T,1)| \geq d_{2,2}\,(n-4)!. \tag{5.7}$$

For $|\mathsf{d}(\mathsf{B}(S,2)) \cap \mathsf{d}(\mathsf{B}(T,1))| \geq d_{2,1}\,(n-4)!$, we consider the set

$$T^* = \{\beta_1 \in \mathsf{d}(T) \mid d(\beta_1,\beta_2) = 1, \text{for some } \beta_2 \in \mathsf{d}(\mathsf{B}(T,1))\}.$$

Evidently, at most $n-1$ elements in $\mathsf{d}(T)$ can be connected to a fixed β_2, whence

$$|T^*| \geq \frac{1}{2} d_{2,1}\,(n-5)!.$$

Let $T_1 \subset T^*$ be some maximal set such that any pair of T_1-vertices (β_1, β_1') has at least distance $d(\beta_1, \beta_1') > 6$. Then $|T_1| > |T^*|/(n-1)^7$ since $|\mathsf{B}(v,6)| < \sum_{i=1}^{6}(n-1)^i < (n-1)^7$. Any two of the paths from $\mathsf{d}(S)$ to $T_1 \subset \mathsf{d}(T)$ are of the form $(\alpha_1,\alpha_2,\beta_2,\beta_1)$ and vertex disjoint since each of them is contained in $\mathsf{T}(\beta_1, 3)$. Accordingly there are a.s. at least

$$\frac{1}{2} d_{2,1}\,(n-5)!/(n-1)^7 \tag{5.8}$$

vertex disjoint paths connecting $\mathsf{d}(S)$ and $\mathsf{d}(T)$. In case of $|\mathsf{d}(\mathsf{B}(S,2)) \cap \mathsf{B}(T,1)| \geq d_{2,2}\,(n-3)!$ we analogously conclude, that there exist a.s. at least

$$d_{2,2}\,(n-4)!/(n-1)^5 \tag{5.9}$$

vertex disjoint paths of the form $(\alpha_1, \alpha_2, \beta_2)$ connecting $\mathsf{d}(S)$ and $\mathsf{d}(T)$.
It remains to consider the case $|\mathsf{B}(S,2)| > \frac{2}{3} \cdot n!$. By construction both: S and T satisfy eq. (5.2), whence we can, without loss of generality assume that also $|\mathsf{B}(S,2)| > \frac{2}{3} \cdot n!$ holds. But then

$$|\mathsf{B}(S,2) \cap \mathsf{B}(T,2)| > \frac{1}{3}\, n!$$

and for each $\alpha_2 \in \mathsf{B}(S,2) \cap \mathsf{B}(T,2)$ we select $\alpha_1 \in \mathsf{d}(S)$ and $\beta_1 \in \mathsf{d}(T)$. We derive in analogy to the previous arguments that there exist a.s. at least

$$d_2\,(n-2)!/(n-1)^5 \tag{5.10}$$

pairwise vertex disjoint paths of the form $(\alpha_1, \alpha_2, \beta_1)$ and the proof of the lemma is complete. □

Proof of Theorem 1. To prove the theorem we employ an argument due to Ajtai *et al.* [1] originally used for n-cubes and independent edge-selection. We proceed along the lines of [38] and select the $\Gamma(S_n, T_n)$-vertices in two distinct randomizations.
Let $x_1, x_2 > 1$ such that $\frac{1}{x_1} + \frac{1}{x_2} = 1$. First we select with probability $\frac{1+\epsilon_n/x_1}{n}$ and second with probability $\frac{\epsilon_n}{x_2 \cdot n}$. The probability of not being chosen in both rounds is given by

$$\left(1 - \frac{1+\epsilon_n/x_1}{n}\right)\left(1 - \frac{\epsilon_n}{x_2 \cdot n}\right) \geq 1 - \frac{1+\epsilon_n}{n},$$

whence it suffices to prove that after the second randomization there exists a giant component with the property $|C_n^{(1)}| \sim |\Gamma_{n,k}|$.
After the first randomization each $\Gamma(S_n, T_n)$-vertex has been selected with probability $\frac{1+\epsilon_n/x_1}{n}$ and according to Lemma 6, we have

$$|\Gamma_{n,k}(x_1)| \sim \wp(\epsilon_n/x_1)\, |\Gamma_n(x_1)| \quad \text{a.s.}, \tag{5.11}$$

where $\Gamma_n(x_1) \subset \Gamma_n$. Suppose $\Gamma_{n,k}(x_1)$ contains a "large" component, S. To be precise a component S of size

$$(n-2)! \leq |S| \leq (1-b)\, |\Gamma_{n,k}(x_1)|, \quad \text{where } b > 0.$$

Then there exists a split of $\Gamma_{n,k}(x_1)$, (S,T), satisfying the assumptions of Lemma 8. We observe that Lemma 4 limits the number of ways these splits can be constructed. Recall (eq. (4.1))

$$M_k(n) = \frac{1}{2^{k+2}} \cdot \left[\frac{1}{4k(k+1)}\right]^k \cdot n^{\frac{2}{3}+k\delta}.$$

Obviously, there are at most $2^{n!/M_k(n)}$ ways to select S of such a split. Now we employ Lemma 8. In view of $(n-2)! \leq |S|$, Lemma 8 implies that there exists some $c > 0$ such that a.s. $d(S)$ is connected to $d(T)$ in $\Gamma(S_n, T_n)$ via at least $c \cdot n!/n^{12} \leq c \cdot |S|/n^{10}$ vertex disjoint paths of length

≤ 3.
We next perform the second randomization and select $\Gamma(S_n, T_n)$-vertices with probability $\frac{\epsilon_n/x_2}{n}$. None of the above $c \cdot |S|/n^{10}$ paths can be selected during this process. Since any two paths are vertex disjoint the expected number of such splits is, by linearity of expectation, less than

$$2^{n!/M_k(n)}(1-(\epsilon_n/x_2 n)^4)^{\frac{c\cdot n!}{n^{12}}} \leq 2^{n!/M_k(n)} e^{-c' n!/n^{16}} \quad \text{for some } c, c' > 0. \tag{5.12}$$

Accordingly, choosing k sufficiently large the expected number of these $\Gamma_{n,k}(x_1)$-splits tends to zero, i.e. for any $k \geq k_0 \in \mathbb{N}$ there exists a.s. no two component split (S, T) of $\Gamma_{n,k}(x_1)$ with the property $\rho_0 |\Gamma_{n,k}(x_1)| = |S| \leq |T|$. Consequently, there exists some subcomponent $C_n(x_1)$ with the property

$$|C_n(x_1)| = |\Gamma_{n,k}(x_1)| \sim \wp(\epsilon_n/x_1)\, |\Gamma(x_1)| \quad \text{a.s.},$$

obtained by the merging of the subcomponents of size $\geq M_k(n)$ generated during the first randomization via the paths selected during the second. Since $\wp(\epsilon_n/x_1)$ is continuous in the parameter ϵ_n/x_1, see eq. (3.1), we derive, for x_1 tending to 1

$$|C_n^{(1)}| = \lim_{x_1 \to 1} |C_n(x_1)| \sim \wp(\epsilon_n)|\Gamma_n| \quad \text{a.s.} \tag{5.13}$$

It remains to prove uniqueness. Any other largest component, $\tilde{C}_n$, is necessarily contained in $\Gamma_{n,k}$. However, we have just proved $|C_n^{(1)}| \sim \wp(\epsilon_n)|\Gamma_n|$ and according to Lemma 6, $\wp(\epsilon_n)|\Gamma_n| \sim |\Gamma_{n,k}|$. Therefore $|\tilde{C}_n| = o(|C_n^{(1)}|)$, whence $C_n^{(1)}$ is unique. □

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