New Results on the Stack Ramification of Binary Trees

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Abstract

The stack-size of a tree T is the number of cells of a stack needed to traverse T in postorder. In this paper we show that the average number of proper subtrees having the same stack-size as the whole tree is asymptotically 1 with a variance of $2+o(1)$. The total number of subtrees with a stack-size one less than that of the whole tree is identical to 2. Counting only maximal subtrees changes this number to $1+o(1)$ with a variance of $o(1)$.

Keywords: analysis of algorithms, combinatorial problems

1 Introduction and Fundamental Definitions

Let $T = (I, L, r)$ be an (extended) binary tree ([Knu73] p. 399) with the set of internal nodes I, the set of leaves L, and the root r. We call $|T| := |I|$ the *size* of T. Choosing a node $v \in I \cup L$ we denote by T_v the subtree of T with the root v. For $v \in I$ we denote by v_l (resp. v_r) the left (resp. right) son of v. Let $r, v_1, v_2, \ldots, v_i$ be the path from the root to node v_i and let p be some predicate defined on $I \cup L$. If $p(v_i) = \mathsf{true} \wedge \forall j \in [1..i[: p(v_j) = \mathsf{false}$ we call T_{v_i} *maximal* w.r.t. the predicate p.
Each expression consisting of brackets, binary operators and operands may be represented by a binary tree (*syntax tree*) where the operands are the labels of the leaves and the internal nodes represent the operators. For example, the expressions $E_1 := x_1/((x_2 - x_3) \uparrow ((x_4 + x_5) * x_6))$ and $E_2 := x_1/((x_2 - x_3) \uparrow x_4 + x_5 * x_6)$ correspond with the trees T_1 and T_2 of Figure 1, respectively.

A well known strategy to evaluate the corresponding expression from its syntax tree is the postorder traversal of the tree using a stack ([Kem84] pp. 130). The *stack-function* $S : I \cup L \to I\!N$ of a binary tree $T = (I, L, r)$ is defined by

$$S(v) := \begin{cases} 1 & : \text{ if } v \in L, \\ \max(S(v_l), S(v_r) + 1) & : \text{ if } v \in I. \end{cases}$$

Figure 1: The syntax trees of the expressions E_1 and E_2. Here, we have assumed that the precedences of the operators $\{+, -, *, /, \uparrow\}$ are $\uparrow > * = / > + = -$.

$S(v)$ is the maximum number of nodes stored in the stack during the postorder traversal of the subtree T_v [BKR72], [Kem84]. For a detailed description of the relation between the postorder traversal of a tree and the resulting stack configurations see [Kem89]. From now on the number $S(v)$ is called the *stack-number* of the node $v \in I \cup L$. For $T \in \mathcal{B}$, $T = (I, L, r)$, we use $S(T)$ as a surrogate for $S(r)$.
In [Kem92] a stack ramification matrix $R^T(n)$ of a binary tree T with n internal nodes was introduced. It reflects the distribution of the stack-numbers appearing in T; all possible values are regarded independent of the stack-number of the root.
There is an optimal algorithm, based on the use of registers, to evaluate an expression represented by a tree T. In this case the so called register-function calculates the minimal number of registers needed. We use $reg(T)$ to denote the value of the register-function of the tree T. The notion of a ramification matrix associated with a binary tree T has been introduced in [Vie87] reflecting the distribution of $reg(T)$.
In [YM94] the Horton-Strahler ordering (an equivalent notion of the register-function) of binary trees was studied but the problem of determining the number of maximal subtrees T_v of T having $reg(T_v) = reg(T) - 1$ was only investigated empirically. This number was needed to calculate the so-called bifurcation ratio near the root, a parameter which is of interest to geologists. Later, H. Prodinger was able to solve this problem giving an asymptotic equivalent of the number in question [Pro97].
In this paper we consider the following (related) parameters of a binary tree $T = (I, L, r)$ with stack-number $p := S(r)$:

- The average number of nodes $v \in I \backslash \{r\}$ having the stack-number p,
- The average number of nodes $v \in I \cup L$ having the stack-number $p - 1$ and
- The average number of maximal subtrees having the stack-number $p - i$, i constant.

Those parameters give a more detailed insight into the stack ramification of binary trees. Since they are related to the stack-number of the root they provide information about the development of the costs for the postorder traversal. In the sequel we denote $\mathcal{B}$ as the family of extended binary trees and we write $\mathcal{B}_n$ for the set of extended binary trees having n internal nodes. Further for $T = (I, L, r) \in \mathcal{B}$ and $i \in \mathbb{N}_0$ we let $\delta(T, i)$ be the number of nodes $v \in (I \cup L) \backslash \{r\}$ having the stack-number $S(r) - i$. We define $\hat{\delta}(T, i)$ to be the number of nodes $v \in I \cup L$ with $S(v) = S(r) - i$ and no predecessor of v has v's stack-number. By $[z^n]F(z)$ we denote the coefficient of z^n in the expansion of $F(z)$ around $z = 0$.
In the following section we derive some average numbers referring to the stack-function as described above. The order in which they are presented is somehow random.

2 The Results

All our observations are based on the following generating function given by R. Kemp in [Kem92]:

Lemma 1 *The ordinary generating function $\hat{S}_p(z)$ for all extended binary trees with stack-number less than or equal to p is given by*

$$\hat{S}_p(z) = 2 \frac{(1 + \sqrt{1 - 4z})^p - (1 - \sqrt{1 - 4z})^p}{(1 + \sqrt{1 - 4z})^{p+1} - (1 - \sqrt{1 - 4z})^{p+1}} .$$

Figure 2: The four cases for deriving $M_p(z,y)$ and $\overline{M}_p(z,y)$.

□

Now, we prove the following

Theorem 1 *The average number of nodes having the root's stack-number is given by*

$$\frac{\sum_{T\in\mathcal{B}_n}\delta(T,0)}{|\mathcal{B}_n|} = 1 - 2n^{-1} + \mathcal{O}(n^{-2})\,.$$

Proof: Let $T\in\mathcal{B}$, $T=(I,L,r)$. To prove Theorem 1 we derive the generating function $\overline{M}_p(z,y) := \sum_{T\in\mathcal{B},\, S(T)=p} y^{\delta(T,0)}z^{|T|}$ which counts those nodes of $I\backslash\{r\}$ by means of y which have the stack-number $p=S(r)$. Using the substitutions $\varepsilon := \sqrt{1-4z}$ and $u := (1-\varepsilon)/(1+\varepsilon)$ (i.e. $z=\frac{u}{(1+u)^2}$) and defining $S_p(z) := \hat{S}_p(z) - \hat{S}_{p-1}(z)$ we get

$$\hat{S}_p(z) = \frac{(1-u^p)(1+u)}{1-u^{p+1}} \quad , \quad S_p(z) = \frac{(u^{p-1}+u^p)}{(1-u^{p+1})}\frac{(u-1)^2}{(1-u^p)}\,. \tag{1}$$

The function $S_p(z)$ is the generating function for those $T\in\mathcal{B}$ with $\mathcal{S}(T)=p$.
Now, let $v\in I$ with $S(v)=p=S(r)$. Figure 2 shows all possible cases for the stack-numbers of the sons of v. Only the cases 2 and 3 lead to a contribution for y in our generating function. We get

$$\begin{aligned}\overline{M}_p(z,y) &= \underbrace{zS_{p-1}^2(z)}_{\text{Case 1}} + \underbrace{zyS_{p-1}(z)\overline{M}_p(z,y)}_{\text{Case 2}} + \underbrace{zy\overline{M}_p(z,y)\hat{S}_{p-2}(z)}_{\text{Case 3}} + \underbrace{zS_{p-1}(z)\hat{S}_{p-2}(z)}_{\text{Case 4}}\\ &= \frac{zS_{p-1}(z)\hat{S}_{p-1}(z)}{1-zy\hat{S}_{p-1}(z)}\,.\end{aligned}$$

To compute the expected value we have to consider $\overline{D}_p(z) := \frac{\partial}{\partial y}\overline{M}_p(z,y)|_{y=1}$. Using (1) and the fact that $\overline{M}(z,y)|_{y=1} = S_p(z)$ we find

$$\begin{aligned}\overline{D}_p(z) &= \frac{z^2S_{p-1}(z)\hat{S}_{p-1}^2(z)}{(1-z\hat{S}_{p-1}(z))^2} = z^2S_{p-1}(z)\hat{S}_{p-1}^2(z)\hat{S}_p^2(z)\\ &= \frac{(u+1)(u-1)^2(u-u^p)u^{p-1}}{(1-u^p)(1-u^{p+1})^2}\\ &= \frac{1-u^2}{u(u^p-1)} + \frac{1-2u^2+u^4}{u^2(u^{p+1}-1)^2} + \frac{1-u-2u^2+u^3+u^4}{u^2(u^{p+1}-1)}.\end{aligned}$$

Note that $\overline{D}_p(z)$ has a singularity of smallest modulus at $z=\frac{1}{4}$, i.e. $u=1$.
We set $\overline{D}(z) := \sum_{p\geq 2}\overline{D}_p(z)$ since we have to consider all possible values of p. Then

$[z^n]\overline{D}(z) = \sum_{T\in\mathcal{B}_n} \delta(T,0)$ holds. Expanding the geometric series of $\overline{D}_p(z)$ in terms of k, splitting up those for $k=0$ and summing over p we further obtain

$$\overline{D}(z) = -u + (u^2 + u^{-2} - 2) \underbrace{\sum_{p\geq 3}\sum_{k\geq 1} k u^{kp}}_{=:\vartheta} . \tag{2}$$

To get an asymptotic for the coefficient at z^n of ϑ we use the Mellin summation formula described in [FGD95], [FlOd90] and [Pro87]. We set $u := e^{-t}$ which is a well suited substitution since $\mathcal{M}(e^{-at}) = \Gamma(s)a^{-s}$ wherein $\mathcal{M}(f)$ denotes the Mellin transform of the function f and $\Gamma(s)$ stands for the complete gamma function (see [AbSt70]). The former identity is only valid if the exponent of e is not zero. That is why we transformed $\overline{D}(z)$ into the representation (2) in which all those terms (induced by $k=0$) are eliminated. Now, we are interested in the behaviour as $t \to 0$ giving the local expansion of $\overline{D}(z)$ around the singularity $u=1$. As described in [FGD95] we have to compute the Mellin transform of our double sum. A short computation gives

$$\mathcal{M}(\sum_{p\geq 3}\sum_{k\geq 1} k u^{pk}) = \Gamma(s)\zeta(s,3)\zeta(s-1), \tag{3}$$

wherein $\zeta(s,i)$ is the Hurwitz zeta function and $\zeta(s) := \zeta(s,1)$ [AbSt70]. The asymptotic behaviour of ϑ is now the sum of the residues of (3) times t^{-s} left to the fundamental strip. We find

$$\frac{\frac{2}{3}\pi^2 - 5}{4t^2} - \frac{1}{2t} + \frac{5}{24} - \frac{1}{48}t^2 + \mathcal{O}(t^3) . \tag{4}$$

The contribution of the other parts of (2) are considered as Taylor expansions around $t=0$. We get

$$\overline{D}(z) \sim -6 + \frac{2}{3}\pi^2 - t + \left(\frac{2}{9}\pi^2 - \frac{4}{3}\right)t^2 - \frac{1}{2}t^3 + \left(\frac{4}{135}\pi^2 - \frac{5}{72}\right)t^4 - \frac{29}{360}t^5 + \mathcal{O}(t^6) .$$

Now we have to resubstitute to get a result in terms of z. Only the coefficients of t and t^3 are of interest since the other ones only lead to a contribution which can be neglected. Thus, we have to consider $-t - \frac{1}{2}t^3$ and since $t \sim 2\sqrt{1-4z} + \frac{2}{3}(1-4z)^{3/2}$, $z \to \frac{1}{4}$, we can use the identities

$$[z^n](1-z)^\alpha = \binom{n-\alpha-1}{n} = \frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)\Gamma(n+1)},$$

$$\Gamma(n-\frac{1}{2}) = (2n-2)!4^{-(n-1)}\sqrt{\pi}/(n-1)!$$

and

$$|\mathcal{B}_n| = \frac{1}{n+1}\binom{2n}{n} = \frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+2)}$$

to obtain

$$\begin{aligned}
\frac{[z^n]\overline{D}(z)}{|\mathcal{B}_n|} &= \frac{4^n\Gamma(n-\frac{1}{2})\Gamma(n+2)}{\sqrt{\pi}\,\Gamma(2n+1)} - \frac{7\cdot 4^n\Gamma(n-\frac{3}{2})\Gamma(n+2)}{2\sqrt{\pi}\,\Gamma(2n+1)} + \mathcal{O}(n^{-2}) \\
&= \frac{2(n+1)}{2n-1} + \frac{28(n-1)(n+1)}{(2n-1)(2n-2)(2n-3)} + \mathcal{O}(n^{-2}) \\
&= \frac{4(n-5)(n+1)}{(2n-3)(2n-1)} + \mathcal{O}(n^{-2}) = 1 - 2n^{-1} + \mathcal{O}(n^{-2}) .
\end{aligned}$$

□

Thus, for large n there exists one subtree on the average which needs the same number of stack-cells for evaluation as the whole tree. Now the variance is of interest.

Theorem 2 *The random variable describing the number of nodes with the same stack-number as the root has the variance*

$$\sigma^2 = \frac{\sum_{T\in\mathcal{B}_n}\delta(T,0)^2}{|\mathcal{B}_n|} - \left(\frac{\sum_{T\in\mathcal{B}_n}\delta(T,0)}{|\mathcal{B}_n|}\right)^2 = 2 - 10n^{-1} + \mathcal{O}(n^{-2}).$$

Proof: To determine the variance we use the second factorial moment which we obtain by means of the generating function $\overline{U}_p(z) := \frac{\partial^2}{\partial y^2}\overline{M}_p(z,y)|_{y=1}$. Using the former results we get

$$\begin{aligned}\overline{U}_p(z) &= \left.\frac{2z^3 S_{p-1}(z)\hat{S}^3_{p-1}(z)}{(1-zy\hat{S}_{p-1}(z))^3}\right|_{y=1} = 2z^3 S_{p-1}(z)\hat{S}^3_{p-1}\hat{S}^3_p(z)\\ &= \frac{2(1-u-u^2+2u^3-u^5)}{u^3(u^{p+1}-1)} + \frac{2(u^2-1)}{u(u^p-1)} + \frac{2(1-3u^2+3u^4-u^6)}{u^3(u^{p+1}-1)^3}\\ &\quad + \frac{2(2-u-5u^2+2u^3+4u^4-u^5-u^6)}{u^3(u^{p+1}-1)^2}.\end{aligned}$$

Again, we must consider all values of p. Thus, we define $\overline{U}(z) := \sum_{p\geq 2}\overline{U}_p(z)$ and by analogous calculations as before we obtain

$$\overline{U}(z) = 2u + \frac{(u-1)^3(u+1)^2}{u^2}\underbrace{\sum_{p\geq 3}\sum_{k\geq 1}(k^2+k)u^{kp}}_{=:\alpha} + \frac{(u-1)^3(u+1)^2}{u^3}\underbrace{\sum_{p\geq 3}\sum_{k\geq 1}(k^2-k)u^{kp}}_{=:\beta}.$$

Again, we use the Mellin summation formula for deriving an asymptotic result. For the two sums we get

$$\mathcal{M}(\alpha) = \Gamma(s)\zeta(s,3)(\zeta(s-2)+\zeta(s-1))$$

and

$$\mathcal{M}(\beta) = \Gamma(s)\zeta(s,3)(\zeta(s-2)-\zeta(s-1)).$$

Taking the other parts into account yields

$$\overline{U}(z) \sim 2 - 2t + (\frac{2}{3}\pi^2 - \frac{10}{3})t^2 - \frac{7}{3}t^3 + (\frac{5}{18}\pi^2 - \frac{187}{180})t^4 + \mathcal{O}(t^5).$$

With the same arguments as before only $-2t - \frac{7}{3}t^3$ is of interest. Resubstituting and using the same identities again gives the final result

$$\frac{[z^n]\overline{U}(z)}{|\mathcal{B}_n|} = \frac{8(n-9)(n+1)}{(2n-3)(2n-1)} + \mathcal{O}(n^{-2}) = 2 - 12n^{-1} + \mathcal{O}(n^{-2}).$$

To compute the variance we use $\sigma^2(X) = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - \mathbb{E}^2[X]$ and obtain

$$2 - 10n^{-1} + \mathcal{O}(n^{-2}).$$

□

In the table of Figure 3 you may find some numerical values of our parameter.

The next step is to count the number of nodes v of $T \in \mathcal{B}$, $T = (I, L, r)$, with $S(v) = S(r) - 1$. We are able to prove the following

Figure 3: Table of exact and asymptotical average values for the number of nodes with the root's stack-number.

Theorem 3 *The average number of nodes having a stack-number one less than the root is given by*

$$\frac{\sum_{T\in\mathcal{B}_n}\delta(T,1)}{|\mathcal{B}_n|}=2\,.$$

Proof: To prove this theorem we derive a generating function for the number in question and show that it is two times the generating function for the extended binary trees.
The family $\mathcal{B}$ has the well known generating function $B(z)=(1-\sqrt{1-4z})/(2z)$. Using the above substitutions yields $\mathcal{B}(z)=1+u$.
Again we use the generating functions $\hat{S}_p(z)$ and $S_p(z)$ from (1) to derive a generating function $M_p(z,y):=\sum_{T\in\mathcal{B},\,S(T)=p}y^{\delta(T,1)}z^{|T|}$ which counts those nodes of $T\in\mathcal{B}$, $T=(I,L,r)$, by means of y which are marked by $\mathcal{S}(r)-1$. To do this we have to consider the cases of Figure 2. We can use our generating function $\overline{M}_p(z,y)$ for the cases in which we have a node v with $S(v)=p-1$ to express the possibility that $S(v_l)=S(v)$. If we translate the above cases into the generating function $M_p(z,y)$ we get:

$$\begin{aligned}
M_p(z,y) &= \underbrace{zy^2\overline{M}_{p-1}^2(z,y)}_{\text{Case 1}}+\underbrace{zy\overline{M}_{p-1}(z,y)M_p(z,y)}_{\text{Case 2}}+\underbrace{zM_p(z,y)\hat{S}_{p-2}(z)}_{\text{Case 3}}\\
&\quad+\underbrace{zy\overline{M}_{p-1}(z,y)\hat{S}_{p-2}(z)}_{\text{Case 4}}\\
&= \frac{zy^2\overline{M}_{p-1}(z,y)+zyM_{p-1}(z,y)\hat{S}_{p-2}(z)}{1-zy\overline{M}_{p-1}(z,y)-z\hat{S}_{p-2}(z)}\,.
\end{aligned}$$

To find the expected value we have to work with $D_p(z):=\frac{\partial}{\partial y}M_p(z,y)|_{y=1}$. Since we are interested in all possible values of p we have to consider $D(z):=\sum_{p\geq 2}D_p(z)$ which yields $[z^n]D(z)=\sum_{T\in\mathcal{B}_n}\delta(T,1)$. Using our representation of $\hat{S}_p(z)$ and $S_p(z)$ and the fact that $\overline{M}_p(z,y)|_{y=1}=M_p(z,y)|_{y=1}=S_p(z)$ we get

$$\begin{aligned}
D_p(z) &= (u^2-1)^2\,\frac{u^{p-2}(u^2+u^p-3u^{p+1}-u^{p+2}+u^{2p+1}+3u^{2p+2}-u^{2p+3}-u^{3p+1})}{(u-u^p)(u^p-1)^2(u^{p+1}-1)^2}\\
&= (u^2-1)^2\Big(\frac{1}{u^2(1-u^p)}+\frac{1}{(1-u^2)(u-u^p)}-\frac{1}{u^2(u^p-1)^2}+\frac{1}{u^2(u^{p+1}-1)^2}\\
&\quad+\frac{u^2-u-1}{u^2-u^4-u^{p+3}+u^{p+5}}\Big)\,.
\end{aligned}$$

Expanding the geometric series in terms of k, splitting up those for $k=0$ and summing over p we further obtain

$$\begin{aligned}
D(z) &= \frac{(1-u^2)}{u}\sum_{p\geq 2}\sum_{k\geq 1}u^{k(p-1)}+(2-u^{-2}-u^2)\sum_{p\geq 2}\sum_{k\geq 1}(k+1)u^{kp}\\
&\quad-(2-u^{-2}-u^2)\sum_{p\geq 2}\sum_{k\geq 1}u^{kp}-(2-u^{-2}-u^2)\sum_{p\geq 2}\sum_{k\geq 1}(k+1)u^{k(p+1)}\\
&\quad-(u^2-u+u^{-1}+u^{-2}-2)\sum_{p\geq 2}\sum_{k\geq 1}u^{k(p+1)}\,.
\end{aligned}$$

Now, shifting the index of summation of p and combining corresponding sums yields

$$\begin{aligned} D(z) &= \frac{(1-u^2)}{u}\left(\frac{1}{(1-u)}-1\right)+\frac{(1-u^2)}{u}\left(\frac{1}{(1-u^2)}-1\right) \\ &\quad -(2-u^{-2}-u^2)\left(\frac{1}{(1-u^2)}-1\right)-(u^2+u^{-2}-2)\left(\frac{1}{(1-u^2)^2}-1\right)=2u\,. \end{aligned}$$

Since the expansion of $\mathcal{B}(z)=1+u$ differs from that one of u only in the term for z^0 our proof is complete. □

One might think that this average number of two nodes having the stack-number of the root minus one is directly connected with that one of Theorem 1. But this is not the case as we will see by the next theorem.

Theorem 4 *The average number of maximal subtrees having a stack-number one less than the root is given by*

$$\frac{\sum_{T\in\mathcal{B}_n}\hat{\delta}(T,1)}{|\mathcal{B}_n|}=1+2n^{-1}+\mathcal{O}(n^{-2})\,.$$

Proof: We first derive the generating function $W_p(x,y)$, similar to the previous one, in which only those nodes are counted by y which are marked with $p-1$ but do not have any predecessor with the same mark. Again, we have to consider the four cases of Figure 2 which yields:

$$W_p(z,y)=\underbrace{zy^2S_{p-1}^2(z)}_{\text{Case 1}}+\underbrace{zyS_{p-1}(z)W_p(z,y)}_{\text{Case 2}}+\underbrace{zW_p(z,y)\hat{S}_{p-2}(z)}_{\text{Case 3}}+\underbrace{zyS_{p-1}(z)\hat{S}_{p-2}(z)}_{\text{Case 4}}\,.$$

By defining $T_p(z):=\frac{\partial}{\partial y}W_p(z,y)|_{y=1}$ and $T(z):=\sum_{p\geq 2}T_p(z)$ we get the required result because $[z^n]T(z)=\sum_{T\in\mathcal{B}_n}\hat{\delta}(T,1)$. By inserting (1) into the above equation for $W_p(x,y)$ and using the identity $W_p(z,y)|_{y=1}=S_p(z)$ we get:

$$\begin{aligned} T_p(z) &= \frac{2zS_{p-1}^2(z)+zS_{p-1}(z)S_p(z)+zS_{p-1}(z)\hat{S}_{p-2}(z)}{1-z\hat{S}_{p-1}(z)} \\ &= (u+1)(u-1)^2\left(\frac{u}{(u+1)(u-u^p)^2}+\frac{2}{(u-1)(u+1)^2(u-u^p)}\right. \\ &\quad \left.+\frac{1}{(u-1)u(u^p-1)}+\frac{1}{u(u+1)(u^{p+1}-1)^2}+\frac{u^2-2u-1}{(u-1)u(u+1)^2(u^{p+1}-1)}\right). \end{aligned} \tag{5}$$

Note that $T_p(z)$ has a singularity of smallest modulus at $z=\frac{1}{4}$, i.e. $u=1$.

We obtain $T(z)$ by expressing the fractions in (5) as geometric series in terms of k and summing over all possible values of p. By splitting up the terms for $k=0$ and shifting the index of summation p we are able to combine all these sums which yields

$$T(z)=-\frac{(u^2-2u-2)u}{(u+1)^2}+\frac{2(u-1)^2}{u}\underbrace{\sum_{p\geq 3}\sum_{k\geq 1}ku^{pk}}_{=\vartheta}\,. \tag{6}$$

Figure 4: Table of exact and asymptotical average values for the number of maximal subtrees having a stack-number one less than the root.

Note that the sum of (6) is the same as the sum of (2). Thus we can reuse our result for the asymptotic behaviour of ϑ as given in (4). The contribution of the other parts of (6) are considered as Taylor expansions around $t = 0$. We get

$$T(z) \sim -\frac{7}{4} + \frac{1}{3}\pi^2 - t + \left(\frac{1}{36}\pi^2 - \frac{11}{48}\right) t^2 + \frac{1}{6}t^3 - \frac{17}{288}t^4 - \frac{1}{288}t^6 + \mathcal{O}(t^7)\,.$$

Now we have to resubstitute to obtain a result in terms of z. Again, only the coefficients of t and t^3 are of interest. Thus we have to consider $-t + \frac{1}{6}t^3$ and since $t \sim 2\sqrt{1-4z} + \frac{2}{3}(1-4z)^{3/2}$, $z \to \frac{1}{4}$, we find

$$\begin{aligned}
\frac{[z^n]T(z)}{|\mathcal{B}_n|} &= \frac{4^n\Gamma(n-\frac{1}{2})\Gamma(n+2)}{\sqrt{\pi}\,\Gamma(2n+1)} + \frac{4^n\Gamma(n-\frac{3}{2})\Gamma(n+2)}{2\sqrt{\pi}\,\Gamma(2n+1)} + \mathcal{O}(n^{-2}) \\
&= \frac{4n(n+1)}{2n(2n-1)} + \frac{4(n-1)n(n+1)}{n(2n-1)(2n-2)(2n-3)} + \mathcal{O}(n^{-2}) \\
&= 1 + \frac{5}{2}\frac{1}{2n-3} + \frac{3}{2}\frac{1}{2n-1} + \mathcal{O}(n^{-2}) = 1 + 2n^{-1} + \mathcal{O}(n^{-2})\,.
\end{aligned}$$

□

Thus, for large n we only have one node v in $T = (I, L, r)$ with $S(v) = S(r) - 1$ and no predecessor v' of v has $S(v') = S(v)$. Some numerical values of our parameter can be found in the table of Figure 4. The next step is to compute the variance in order to see how close to the truth the average is.

Theorem 5 *The random variable describing the number of maximal subtrees with a stack number one less than the root has the variance*

$$\sigma^2 = \frac{\sum_{T\in\mathcal{B}_n}\hat{\delta}(T,1)^2}{|\mathcal{B}_n|} - \left(\frac{\sum_{T\in\mathcal{B}_n}\hat{\delta}(T,1)}{|\mathcal{B}_n|}\right)^2 = 2n^{-1} + \mathcal{O}(n^{-2})\,.$$

Proof: We define the generating function for the second factorial moment as $U_p(z) := \frac{\partial^2}{\partial y^2}W_p(z,y)|_{y=1}$.
From the former results we get

$$\begin{aligned}
U_p(z) &= \frac{2zS_{p-1}^2(z) + 2zS_{p-1}(z)T_p(z)}{1 - z\hat{S}_{p-1}(z)} \\
&= \frac{2(u-1)^4u^{2p}(u+1)^2(u^p-1)^2}{(u-u^p)^3(1-u^{p+1})^3}\,.
\end{aligned}$$

Similar calculations as before yield for the summation over all possible values of p

$$U(z) := \sum_{p\geq 2} U_p(z) = -2\frac{u(4u^3-5u^2-6u-1)}{(u+1)^4} - 2\frac{(u-1)^2(1-4u+u^2)}{(u+1)^2u}\underbrace{\sum_{p\geq 3}\sum_{k\geq 1} ku^{kp}}_{=\vartheta}\,.$$

Figure 5: The cases for deriving $W_p(i,z,y)$.

Again we use the Mellin summation technique for deriving an asymptotic result. The sums ϑ are the same as ϑ in (2); so we can reuse (3) and (4). Taking the other parts into account yields

$$U(z) \sim \frac{1}{6}\pi^2 - \frac{1}{4} - \left(\frac{1}{12} + \frac{1}{9}\pi^2\right) t^2 + \frac{2}{3}t^3 - \frac{1}{288}t^4 + \mathcal{O}(t^6)\,.$$

In this case only the contribution of t^3 is of interest. Using the same identities as before gives the final result

$$\frac{[z^n]U(z)}{|\mathcal{B}_n|} = \frac{4^{n+1}\Gamma(n-\frac{3}{2})\Gamma(n+2)}{\sqrt{\pi}\,\Gamma(2n+1)} + \mathcal{O}(n^{-2}) = 4n^{-1} + \mathcal{O}(n^{-2})\,.$$

This implies the variance to be

$$2n^{-1} + \mathcal{O}(n^{-2})\,.$$

□

The variance being 0 for large n implies the average number to be "the reality" for every sequence. Thus, almost every large tree $T = (I, L, r)$ has one maximal subtree T_v with $S(v) = S(r) - 1$. This means that the result of Theorem 3 is implied by nodes which are in a predecessor-successor-relation.

Now we want to consider the average number of maximal subtrees T_v of $T = (I, L, r)$ having $S(v) = S(r) - i$, $i \in I\!N$. We first derive the corresponding generating function without being able to calculate a closed form solution. Afterwards we use this generating function to see that the behaviour for $i = 2$ is also ~ 1.

Let $W_p(i,z,y)$ be the generating function fulfilling $[z^n][y^j]W_p(i,z,y) = |\{T \in \mathcal{B}_n | \mathcal{S}(T) = p \wedge \hat{\delta}(T,i) = j\}|$. Translating the situation of Figure 5 into this generating function yields

$$\begin{aligned} W_p(i,z,y) = \\ & \underbrace{zW_{p-1}(i-1,z,y)\sum_{k=0}^{i-1} W_{p-k}(i-k,z,y)}_{\text{Case 1}} + \underbrace{zyW_{p-1}(i-1,z,y)S_{p-i}(z)}_{\text{Case 2}} \\ & + \underbrace{zW_{p-1}(i-1,z,y)\hat{S}_{p-i-1}(z)}_{\text{Case 3}} + \underbrace{zW_p(i,z,y)\sum_{k=2}^{i-1} W_{p-k}(i-k,z,y)}_{\text{Case 4}} \\ & + \underbrace{zyW_p(i,z,y)S_{p-i}(z)}_{\text{Case 5}} + \underbrace{zW_p(i,z,y)\hat{S}_{p-i-1}(z)}_{\text{Case 6}} \\ = \; & (W_p(i,z,y) + W_{p-1}(i-1,z,x))\, z \left(yS_{p-i}(z) + \sum_{k=2}^{i-1} W_{p-k}(i-k,z,y) + \hat{S}_{p-i-1}(z) \right) \end{aligned}$$

$$
\begin{aligned}
&+zW_{p-1}(i-1,z,y)W_p(i,z,y)+zW_{p-1}^2(i-1,z,y)\\
=\ &\frac{zW_{p-1}(i-1,z,y)\left(\sum_{k=1}^{i-1}W_{p-k}(i-k,z,y)+yS_{p-i}(z)+\hat{S}_{p-i-1}(z)\right)}{1-z\left(\sum_{k=1}^{i-1}W_{p-k}(i-k,z,y)+yS_{p-i}(z)+\hat{S}_{p-i-1}(z)\right)}
\end{aligned}
$$

with

$$W_p(1,z,y)=\frac{zy^2S_{p-1}^2(z)+zyS_{p-1}(z)\hat{S}_{p-2}(z)}{1-zyS_{p-1}(z)-z\hat{S}_{p-2}(z)}.$$

Again, $T_p(i,z):=\frac{\partial}{\partial y}W_p(i,z,y)|_{y=1}$ and we have

$$T_p(i,z)=z\hat{S}_p(z)\left[T_{p-1}(i-1,z)\hat{S}_{p-1}(z)+(S_p(z)+S_{p-1}(z))\left(S_{p-i}(z)+\sum_{k=1}^{i-1}T_{p-k}(i-k,z)\right)\right] \tag{7}$$

with $T_p(1,z)=T_p(z)$. Since 1 is the smallest value for a stack-number our summation over all possible values of p must start with $i+1$. Thus, the desired generating function is $T(z):=\sum_{p\geq i+1}T_p(i,z)$. We tried hard to eliminate the full history, e.g. by differencing [GrKn82] but no attempt gave a manageable solution. However, in order to get an insight of the behaviour for $i>1$, we consider the case $i=2$ and get the

Theorem 6 *The average number of maximal subtrees having a stack-number two less than the root is given by*

$$\frac{\sum_{T\in\mathcal{B}_n}\hat{\delta}(T,2)}{|\mathcal{B}_n|}=1+6n^{-1}+\mathcal{O}(n^{-2}).$$

Proof: By setting $i=2$ in (7) we obtain

$$T_p(2,z)=z\hat{S}_p(z)\left(T_{p-1}(1,z)(\hat{S}_p(z)+S_{p-1}(z))+S_{p-1}(z)S_{p-2}(z)+S_p(z)S_{p-2}(z)\right).$$

Inserting (1) and using $T_p(z)=T_p(1,z)$ yields

$$
\begin{aligned}
&\frac{T_p(2,z)}{(u-1)^2(u+1)u^{p-1}}=\\
&\frac{(u^4-u^{2p}+2u^{p+1}-3u^{p+2}-3u^{p+3}+4u^{2p+2}+4u^{2p+3}-u^{2p+5}-3u^{3p+2}-3u^{3p+3}+2u^{3p+4}+u^{4p+1})}{(u^p-1)^2(u^p-u^2)^2(u^{p+1}-1)^2}.
\end{aligned}
$$

Analogous calculations as before yield

$$
\begin{aligned}
T(2,z)\ =\ &-\frac{u^2(4u^8+7u^7-9u^6-52u^5-90u^4-89u^3-54u^2-19u-3)}{(u+1)^2(u^2+u+1)^4}\\
&+2\frac{(u-1)^2(2+3u+2u^2)}{(u^2+u+1)^2}\underbrace{\sum_{p\geq 4}\sum_{k\geq 1}ku^{kp}}_{=:\omega}.
\end{aligned} \tag{8}
$$

For $u:=e^{-t}$ we have $\mathcal{M}(\omega)=\Gamma(s)\zeta(s,4)\zeta(s-1)$ and ω behaves as

$$\omega\sim\frac{6\pi^2-49}{36t^2}-\frac{1}{2t}+\frac{7}{24}-\frac{7}{120}t^2+\mathcal{O}(t^3),t\to 0.$$

Figure 6: Table of exact and asymptotical average values for the number of maximal subtrees having a stack-number two less than the root.

Considering the other parts of (8) as Taylor expansions around $t = 0$ yields

$$T(2, z) \sim \frac{7}{27}\pi^2 - \frac{127}{108} - t + \left(\frac{121}{1296} - \frac{25}{324}\pi^2\right) t^2 + \frac{5}{6}t^3 + \frac{7}{480}t^4 + \frac{35}{1296}t^6 + \mathcal{O}(t^7)\,.$$

Again only the contributions of t and t^3 are of interest. The same calculations as before end in

$$\begin{aligned} \frac{[z^n]T(2, z)}{|\mathcal{B}_n|} &= \frac{4^n\Gamma(n - \frac{1}{2})\Gamma(n + 2)}{\sqrt{\pi}\;\Gamma(2n + 1)} + \frac{9}{2}\frac{4^n\Gamma(n - \frac{3}{2})\Gamma(n + 2)}{\sqrt{\pi}\;\Gamma(2n + 1)} + \mathcal{O}(n^{-2}) \\ &= 1 + \frac{24n + 9}{4n^2 - 8n + 3} + \mathcal{O}(n^{-2}) = 1 + 6n^{-1} + \mathcal{O}(n^{-2})\,. \end{aligned}$$

□

Thus, the average number of maximal subtrees having the stack-number of the root minus 2 is also asymptotically 1. We have the conjecture that this is always the case for constant i. This conjecture is based on the observation that a different behaviour was only discovered for values of i making $\mathcal{S}(t) - i$ small (which means to operate near the leaves of $T \in \mathcal{B}$). Since this is impossible for $n \to \infty$ and i constant, we believe in our conjecture.
Finally, you find some exact values of $\hat{\delta}(T, 2)$ and our asymptotical equivalent in the table of Figure 6.

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