Advanced Algorithmics

Strategies for Tackling Hard Problems

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5.7 Dual LPs & The Primal-Dual Schema

Starting with an original ("primal") LP, how can we bound on its optimal objective value?

↳ upper bound → find feasible solution

lower bound → ?

$$
\begin{array}{lrl}
\min & 7x_1 + x_2 + 5x_3 & \\
\text{s.t.} & y_1 \cdot (x_1 - x_2 + 3x_3) \geq 10 & y_1 \\
& y_2 \cdot (5x_1 + 2x_2 - x_3) \geq 6 & y_2 \\
& x_1, x_2, x_3 \geq 0 &
\end{array}
$$

$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq \underline{10}$

$7x_1 + x_2 + 5_3 \quad \ngeq$

$\quad \not\geq 2x_2$

$7x_1 + x_2 + 5x_3 \geq 6x_1 + x_2 + 2x_2 \geq \underline{16}$

$7x_1 + x_2 + 5x_3 \overset{!}{\geq} y_1(x_1 - x_2 + 3x_3)$

$\quad + y_2(5x_1 + 2x_2 - x_3)$

$\quad \geq 10y_1 + 6y_2$

Dual:

$\max \quad 10y_1 + 6y_2$

$$
\begin{array}{lrl}
\text{s.t.} & y_1 + 5y_2 & \leq 7 \\
& -y_1 + 2y_2 & \leq 1 \\
& 3y_1 - y_2 & \leq 5 \\
& y_1, y_2 & \geq 0
\end{array}
$$

Optimal solution:
$x^* = (1.75, 0, 2.75)$ with $c^T x^* = 26$.

Dual LPs

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \quad ② \\ & x \geq 0 \end{aligned} \qquad \begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T \cancel{y} \leq c \quad ① \\ & y \geq 0 \end{aligned}$$

$(A^T y)^T = y^T A \leq c^T$

$c = (7, 1, 5)$

$A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$

Generalizations:

- ith constraint in primal with $\geq$ $\leftrightsquigarrow$ $y_i \geq 0$
- ith constraint in primal with $=$ $\leftrightsquigarrow$ y_i unconstrained

Lemma 5.29 (Weak Duality)

If x and y are feasible solutions for the primal resp. dual LP, it holds that $\underline{c^T x} \geq \underline{b^T y}$

Proof: $\underset{\substack{\Uparrow \\ x \geq 0}}{c^T x} \underset{①}{\geq} y^T(Ax) \underset{②}{\geq} y^T b = b^T y$ $\square$.

Duality Theory

Indeed one can show by a closer study that the optimal objective values *match*.

Theorem 5.30 (Strong duality)

The primal LP has a finite optimal objective if and only if the dual has. If x^* resp. y^* are two optimal solutions to the primal resp. dual LP then $c^T \cdot x^* = b^T \cdot y^*$ holds.

Theorem 5.31 (Complementary Slackness Conditions (CSC))

Let x and y be feasible solutions to the primal and dual LP. // none infeasible
The pair (x, y) is optimal *if and only if*

1. $x_j = 0$ or $\sum_{1 \le i \le m} a_{i,j} \cdot y_i = c_j$ for all $1 \le j \le n$, and
2. $y_i = 0$ or $\sum_{1 \le j \le n} a_{i,j} \cdot x_j = b_i$ for all $1 \le i \le m$.

Remark 5.32

1. Strong duality implies that the LP threshold decision problem is in $\mathcal{NP} \cap$ co-$\mathcal{NP}$:
   Yes-certificate: feasible solution; No-certificate: feasible solution *for the dual*.
2. For ILPs, we only get weak duality.

Set Cover LP and its dual

$V(u) = \{j : u \in S_j\}$ $\quad S = \{S_1, \dots, S_k\}$ $\quad S_j \subseteq [n]$

$$\begin{aligned} \min \quad & \sum_{j=1}^{k} c(S_j) \cdot x_j \\ \text{s. t.} \quad & \sum_{j \in V(u)} x_j \geq 1 \quad \forall u \in U \\ & \underline{x \geq 0} \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} \max \quad & \sum_{u \in U} y_u \\ \text{s. t.} \quad & \sum_{u \in S_j} y_u \leq c(S_j) \quad \forall j \in [k] \\ & y \geq 0 \end{aligned}$$

automatic by rules

Intuition:

Pack as much (y_u) of good u as possible, so that for group S_j its capacity $c(S_j)$ is exceeded.

Analysis of greedySetCover by dual fitting

procedure greedySetCover(n, S, c)
  $\mathcal{C} = \emptyset, C = \emptyset$
  // For analysis $j = 1$
  while $C \neq [n]$
    $j^* = \arg\min_{i \in [n]} \frac{c(S_i)}{|S_i \setminus C|}$
    Add j^* to $\mathcal{C}$
    $C = C \cup S_{j^*}$
    // For analysis: $\alpha_j = \frac{c(S_{j^*})}{|S_{j^*} \setminus C|}$; $j = j + 1$
    // For analysis: for $u \in S_{j^*} \setminus C$ *set* $price(u) = \alpha_j$
  return $\mathcal{C}$

price(u) is essentially a dual variable y_u

BUT not directly dual-feasible

$\sum_{u \in U} price(u) = c(\mathcal{C})$ ⇝ y was dual-feasible

⇒ $\mathcal{C}$ was optimal

Lemma 5.33

$y_u = price(u)/H_n$ is dual-feasible.

Proof: $e_1, e_2, \ldots, e_n$ in the order they are covered by greedy Set Cover

$|S_j| = \ell$ ⇝ when e_i was covered, S_j contains $\geq \ell-(i-1)$ uncovered elem.

⇒ S_j could cover e_i (assuming $e_i \in S_j$) with per elem cost $\leq \frac{c(S_j)}{\ell - i + 1}$

↳ also chooses most cost-effective set S_{j^*} ⇝ $\alpha_j \leq \frac{c(S_j)}{\ell - i + 1}$

$$y_{e_i} \leq \frac{1}{H_n} \cdot \frac{c(S_j)}{\ell - i + 1}$$

Consider constraint for S_j

$$\sum_{u \in S_j} y_u = \sum_{m=1}^{\ell} y_{e_{i_m}} \leq \frac{c(S_j)}{H_n}\left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{\ell}\right)$$

$$= \frac{H_\ell}{H_n} \cdot c(S_j) \underset{\ell \leq n}{\leq} c(S_j) \quad \rightsquigarrow \text{dual constraint} \checkmark$$

$\square$

↳ Can easily H_n-approx.:

$$c(\mathcal{C}) = \sum_{u \in \mathcal{U}} \text{price}(u) = H_n \underbrace{\sum_{u \in \mathcal{U}} y_u}_{\text{dual objective}} \underset{\text{weak duality}}{\leq} H_n \cdot OPT_{frac} \leq H_n \cdot OPT$$

Integrality Gap of Set Cover

for minimization problem

Previous result shows that integrality gap $\frac{OPT}{OPT_{frac}} \leq H_n$.

Can we give a lower bound?

Theorem 5.34 (Integrality Gap of Set Cover)

For the set cover ILP and its relaxation holds

$H_n \sim \ln(n)$

$$\underbrace{\frac{OPT}{OPT_{frac}} \geq \frac{\log_2(n+1)}{2\frac{n}{n+1}}}_{\text{show below}} \sim \frac{1}{2\ln 2} H_n \approx 0.721 H_n$$

$\Rightarrow$ essentially not possible to improve greedy algorithm by any LP relaxation-based method

Proof: $n = 2^\ell - 1 \quad \ell \in \mathbb{N} \quad \rightsquigarrow \quad \mathcal{U} = [1 \ldots n] \cong \ell$-bit binary numbers

view $i \in \mathcal{U}$ as a vector $\vec{i} \in \{0,1\}^\ell$ using $(i \bmod n)_2$ (and set $\vec{n} = \vec{0}$)

$S_j = \{ i : \vec{i}^T \vec{j} = 1 \pmod 2 \}$

$j \in [n]$ $\quad c(S_j) = 1 \quad \mathbb{Z}_2$

$$\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 7 \\ \hline 0 & 1 & 0 & = 1 \end{array}$$

$\vec{2}^T \vec{7} = 1$

Can check $|S_j| = \frac{n+1}{2}$ $\quad |V(i)| = \frac{n+1}{2}$

each value i contained in $\frac{n+1}{2}$ sets

$\Rightarrow$ ① $x_j = \frac{2}{n+1}$ $\Rightarrow$ fractional set cover

with $OPT_{frac} = n \cdot \frac{2}{n+1}$

② We need ℓ sets to cover $\mathcal{U}$

$p < \ell$, sets $S_{i_1}, \ldots, S_{i_p}$

$$A = \begin{pmatrix} - & \vec{i_1} & - \\ & \vdots & \\ - & \vec{i_p} & - \end{pmatrix}$$

$p \times \ell$ matrix

$\Rightarrow$ rank $\leq p < \ell$

$\Rightarrow$ null space $(A) \neq \{0\}$

$\Rightarrow \exists j : A \cdot \vec{j} = \vec{0}$

$\Rightarrow j \notin S_{i_1}, \ldots, S_{i_p}$ $\Rightarrow$ not a set cover $\Rightarrow$ $OPT \geq \ell = \log_2(n+1)$ $\square$

Primal-Dual Schema

So far:

- ad hoc methods, a posteriori justified by LP arguments
- rounding algorithms, must solve primal LP to optimality (expensive!)

Can we use duality more directly? Yes!

Idea: Start with two solution; dual feasible, primal is infeasible, but integral

Iteratively: Update solutions to match more of CSC conditions

↳ all satisfied exactly → optimal

for hard problems not be possible efficiently

⇒ relax CSC's a bit, so that

- x remains feasible & integral
- y require relaxed form of CSC

↳ y only gives some lower bound

CSC for set cover

Complementary Slackness Conditions for Set Cover

$$x_j = 0 \ \vee \ \sum_{u \in S_j} y_u = c(S_j) \qquad \forall j \in [k] \qquad \text{(CSC 1)}$$

can be relaxed

$$y_u = 0 \ \vee \ \sum_{j \in V(u)} x_j = 1 \qquad \forall u \in U \qquad \text{(CSC 2)}$$

important

Problem: In general only simultaneously fulfilled by fractional solutions
Relax dual ~~constraints~~ CSCs to

$$y_u = 0 \ \vee \ \sum_{j \in V(u)} x_j \leq f \checkmark \quad \forall u \in U$$

i.e., every element at most f times $\rightsquigarrow$ trivially fulfilled.

Ideas a) Increase y_u until (CSC 1) fulfilled for one set S_j with currently $x_j = 0$
$\rightarrow x_j = 1$

b) $c(S_j) - \sum_{u \in S_j} y_u$ minimal (but not 0)

Primal Dual Set Cover

```
procedure primalDualSetCover(n,S,c)
    f = global frequency
    x⃗ = 0⃗, y⃗ = 0⃗, T = [n]
    while T ≠ ∅
        Choose u ∈ T arbitrary
        Increase y_u until CSC holds for (at least) one more set S_j
        for all S_j with ∑_{u∈S_j} y_u = c(S_j)
            T = T \ S_j
            x_j = 1
    return {j ∈ [k] : x_j = 1}
```

compute $\delta = \min_{j : x_j = 0} c(S_j) - \sum_{u \in S_j} y_u$

✓ $y_u := y_u + \delta$

Theorem 5.35

primalDualSetCover is an f-approximation for SET-COVER.

Proof: $\mathcal{C} = \{ j \in [k] : x_j = 1 \}$ is a set cover ✓ $\rightarrow$ x feasible

Additionally for j with $x_j = 1$ holds $\sum_{u \in S_j} y_u = c(S_j)$ (CSC1)

holds when x_j was set to 1, all y_u with $u \in S_j$ never touched again

$$\sum_{j=1}^{k} c(S_j)\cdot x_j \underset{(csc1)}{=} \sum_{j=1}^{k} x_j \sum_{u\in S_j} y_u$$

$$\| $$

$$c(\mathcal{C})$$

$$= \sum_{u\in \mathcal{U}} y_u \cdot \underbrace{\sum_{j\in V(u)} x_j}_{\leq f} \qquad |V(u)| \leq f$$

$$\leq f \underbrace{\sum_{u\in \mathcal{U}} y_u}_{\text{dual objective}}$$

$$\underset{\substack{y \text{ feasible} \\ \text{weak duality}}}{\leq} f\cdot OPT_{frac}$$

$$\leq f\cdot OPT \qquad \square$$

5.8 Arbitrarily Good Approximations

Goal: Prove some problems inapproximable ⇝ need to define what approximability means

Definition 5.36

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ an optimization problem.

An algorithm A is called *polynomial time approximation scheme (PTAS)* for U, if A computes for each pair $(x, \varepsilon) \in L_I \times \mathbb{R}^+$ a feasible solution which is at most a factor $(1 + \varepsilon)$ worse than the optimum (i.e., ε is the relative error) and needs a polynomial time in $|x|$ (i.e., $\mathcal{O}(|x|^{exp(1/\varepsilon)})$ is possible). ⇒ approximable to any desired error.

If the running time of A is polynomially bounded in $|x|$ and ε^{-1}, A is called a *fully polynomial time approximation scheme (FPTAS)* for U. ◀

In practice, we would like to have FPTAS

Definition 5.37 ∃c : ∃ c-approx. poly-time

$APX = \{U \in \mathcal{NPO} \mid \exists$polynomial time c-approximation algorithm for U, c constant$\}$.

Vertex-Cover ∈ APX

Set-Cover maybe not APX