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5th Exercise sheet for Advanced Algorithmics, Summer 17

Hand In: *Until Wednesday, 31.05.2017, 12:00 am, hand-in box in 48-4 or via email.*

Problem 10

15 points

Prove that it is impossible to perfectly simulate a roll of a fair 6-sided die using random bits in finite worst-case time, i.e., with $t = \mathit{time}_A < \infty$.

Problem 11

15 points

Prove that the set $C = \{000, 111\}^\omega$, i.e., the set of infinite bit sequences on with `dieRoll` does *not* terminate is in the σ-algebra $\mathcal{F}$ generated by the cylinder sets $\pi_x = \{xy : y \in \{0,1\}^\omega\} \subseteq \{0,1\}^\omega$ for $x \in \{0,1\}^\star$. Show, by computing along the construction for C, that $\Pr[C] = 0$ in the probability measure induced by $\Pr[\pi_w] = 2^{-|w|}$.

Problem 12

10 points

Let $P \overset{\mathcal{D}}{=} \mathcal{U}(0,1)$ be a random variable uniformly distributed in $(0,1)$ and let X be a random variable with a Bernoulli $\mathcal{B}(p)$ distribution *conditional* on $P = p$. We also write this as $X \overset{\mathcal{D}}{=} \mathcal{B}(P)$. Compute $\mathbb{E}[X]$.

Problem 13

20 + 30 points

We consider the two definitions for Las Vegas algorithms.

a) Prove Theorem 4.2 from class:

Every Las Vegas algorithm A for $f : \Sigma^\star \to \Gamma^\star$ can be transformed into a randomized algorithm B for f so that for all $x \in \Sigma^\star$ holds

(i) $\Pr[B(x) = f(x)] \;=\; 1$ (always correct)

(ii) $\mathbb{E}\text{-}\mathit{time}_B(x) \;\leq\; 2 \cdot \mathit{time}_A(x)$

b) Prove Theorem 4.3 from class:

Every randomized algorithm B for $f : \Sigma^\star \to \Gamma^\star$ with $\Pr[B(x) = f(x)] = 1$ can be transformed into a Las Vegas algorithm A for f so that for all $x \in \Sigma^\star$ holds

$$\mathit{time}_A(x) \;\leq\; 2 \cdot \mathbb{E}\text{-}\mathit{time}_B(x).$$

Hint: Recall the *Markov's inequality*.

Problem 14

20 + 30 points

Let us consider the model of flipping a fair coin n times and denote by $X \in [0 : n]$ the total number of "heads" among the n coin flips.

a) For the concrete value $n = 100$, compute

(i) the exact probability $\Pr[X \geq 66]$ (use computer algebra!),

(ii) an upper bound for $\Pr[X \geq 66]$ using *Markov's inequality*,

(iii) an upper bound for $\Pr[X \geq 66]$ using *Chebychev's inequality*, (recall the formula for $\mathrm{Var}[X]$), and

(iv) an upper bound for $\Pr[X \geq 66]$ using the *Chernoff bound* for the binomial distribution.

b) Prove that we have for any $\varepsilon > 0$ that $X = \mathbb{E}[X] \pm \mathcal{O}(n^{1/2+\varepsilon})$ w. h. p. as $n \to \infty$.