TECHNISCHE UNIVERSITÄT
KAISERSLAUTERN

Department of Computer Science
Sebastian Wild

Issue Date: 03.05.2015
Version: 2017-05-10 14:21

2nd Exercise Sheet for Advanced Algorithmics, Summer 17

Hand In: *Until Wednesday, 10.05.2015, 12:00 am, hand-in box in 48-4 or via email.*

Problem 3

30 + 30 points

For each of the following problems, either sketch a pseudopolynomial algorithm for solving the problem (proving that the problem is only weakly $\mathcal{NP}$-hard), or show that the problem is strongly $\mathcal{NP}$-hard.

For integer-input problems with numbers in $\mathbb{Z}$, MaxInt of an instance is the largest *absolute value* occurring in the input.

a) Subset Sum

Given: $x_1, \ldots, x_n \in \mathbb{Z}$.

Question: $\exists I \subseteq [n] \; : \; I \neq \emptyset \; \wedge \; \sum_{i \in I} x_i = 0$?

b) 0/1 Integer Programming

Given: Integer linear program (ILP): $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$

Question: Is there $x \in \{0,1\}^n$ with $Ax \leq b$ and $c^T x \geq k$?

Problem 4

15 + 30 + 5 + 15 + 15 + 20 + 10 points

In this exercise, you will show that p-WSAT(CNF$^+$), another variant of weighted satisfiability, is $\mathcal{W}[1]$-hard. The problems is an important reduction partner (see below); it is formally given as follows:

p-WSAT(CNF$^+$):

Given: A formula φ in *positive CNF*, i.e., without negative literals, over the variables $x_1, \ldots, x_n$, and an integer $k \in [n]$.

Parameter: k

Question: Is there a satisfying assignment with weight k, i.e., a vector $\vec{v} \in \{0,1\}^n$ with $v_1 + \cdots + v_n = k$, so that setting $x_i := v_i$ satisfies φ?

(p-WSAT(CNF$^+$) is actually $\mathcal{W}[2]$-complete, but we will not use this result).

To keep you motivated to study yet another peculiar variant of SAT, we do not start with the proof right away, but first *use* the hardness of p-WSAT(CNF$^+$) in two classical graph problems. In both cases, the $\mathcal{W}[1]$-hard problems we have seen so far seem to be of little help in establishing hardness by a simple and direct fpt-reduction.

a) Show p-WSAT(CNF$^+$) $\leq_{fpt}$ p-Hitting-Set, thereby showing that p-Hitting-Set is probably *not* fixed-parameter tractable.

   Here, hitting set is a generalization of the vertex cover problem to hypergraphs:

   p-Hitting-Set:

   Given: a collection of subsets $X_1, \ldots, X_m \subseteq [n]$, integer $k \in \mathbb{N}$.

   Parameter: k

   Question: $\exists I \subset [n] \;:\; |I| = k \;\wedge\; \forall j \in [m] : X_j \cap I \neq \emptyset$?

b) Recall the definition of parametrized dominating set problem:

   p-Dominating-Set:

   Given: graph $G = (V, E)$ and $k \in \mathbb{N}$.

   Parameter: k

   Question: $\exists V' \subset V \;:\; |V'| \leq k \;\wedge\; \forall v \in V : \big(v \in V' \;\vee\; \exists u \in N(v) : u \in V'\big)$?

   Show that p-Dominating-Set is $\mathcal{W}[1]$-hard.

We now turn to the proof that p-WSAT(CNF$^+$) is $\mathcal{W}[1]$-hard; we first construct two handy generic formulæ β and γ that we will need below.

We say that a formula is *k-satisfiable* if it is satisfied by an assignment with weight k, and two formulæ φ and ψ are *k-equivalent* if any weight-k-assignment fulfills φ iff it fulfills ψ.

c) Assume we are given variables $Z_{i,j}$ with $i \in [m]$ and $j \in [n_i]$ that are partitioned into subsets $Z_i = \{Z_{i,1}, \ldots, Z_{i,n_i}\}$.

We only consider assignments with *weight exactly* m. Construct a formula $\beta = \beta(Z_1, \ldots, Z_m)$ in *positive CNF* that is m-equivalent to the statement that v sets *exactly one* variable from each Z_i to true.

Argue that the formula can be computed in poly-time given the variables $Z_{i,j}$.

Hint: Use a pigeon-hole argument for the m true variables.

d) Assume now we are given two groups of variables $X = \{X_1, \ldots, X_n\}$ and $Y = \{Y_1, \ldots, Y_m\}$ and two special variables $X_i \in X$ and $Y_j \in Y$. Show that there is a formula $\gamma(X_i, Y_j)$ that is a *disjunction of positive literals* so that

$$\beta(X,Y) \wedge \gamma(X_i, Y_j) \qquad \text{is 2-equivalent to} \qquad \beta(X,Y) \wedge (X_i \to Y_j).$$

We are now approaching our goal: reducing p-WSAT(2-CNF$^-$), which is $\mathcal{W}[1]$-complete as stated in class, to p-WSAT(CNF$^+$). To fix notation, we transform an instance (φ, k) with φ in negative 2-CNF to an instance (φ', k') of p-WSAT(CNF$^+$), i.e., with φ' in positive CNF.

The crux of the matter is to express the negated literals in φ positively; unfortunately this requires a tricky implicit representation of the original variables $X_1, \ldots, X_n$ in φ:

φ' will have variables $T_{i,j}$ for $i \in [k]$, $j \in [n]$ and $B_{i,j,j'}$ for $i \in [k-1]$, $1 \le j < j' \le n$, with the intended meaning that

$$T_{i,j} \iff v_j = 1 \wedge v_1 + \cdots + v_j = i, \tag{1}$$
$$B_{i,j,j'} \iff v_j = 1 \wedge v_{j'} = 1 \wedge v_1 + \cdots + v_j = i \wedge v_1 + \cdots + v_{j'} = i+1. \tag{2}$$

where $\vec{v} \in \{0,1\}^n$ is the assignment for $X_1, \ldots, X_n$; intuitively $T_{i,j}$ is true iff the ith true variable is X_j and $B_{i,j,j'}$ is true iff the ith true variable is X_j *and* the $(i+1)$st true variable is $X_{j'}$.

For convenient use of our patterns β and γ, we define the groups of variables $T_i = \{T_{i,1}, \ldots, T_{i,}\}$ for $i \in [k]$ and $B_i = \{B_{i,1,2}, \ldots, B_{i,n-1,n}\}$ for $i \in [k-1]$.

e) Construct a formula α in positive CNF that is true iff there is a $\vec{v} \in \{0,1\}^n$ so that Equations (1) and (2) hold for this assignment of new variables ($T_{i,j}$ and $B_{i,j,j'}$).

f) We can now define a positive disjunction $\xi(j)$ that will take the place of $\neg X_j$; precisely:

Construct a formula $\xi(j)$ that is a disjunction of certain variables (from $T_{i,j}$ and $B_{i,j,j'}$), so that $\alpha \wedge \xi(j)$ holds iff Equations (1) and (2) are fulfilled for a $\vec{v}$ with $v_1 + \cdots + v_n = k$ and $v_j = 0$.

g) Show that p-WSAT(2CNF$^-$) $\le_{fpt}$ p-WSAT(CNF$^+$).