Exercise 5

Problem 13

DDDP

We have: multisets A, B, C with $S = \sum_{a \in A} a = \sum_{b \in B} b = \sum_{c \in C} c$

We want: yes/no, whether permutations π_A, π_B, π_C exist such that

(a) $C = Dist(Pos(\pi_A) \cup Pos(\pi_B))$

(b) $Pos(\pi_A) \cap Pos(\pi_B) = \{0, S\}$

3-partition

We have: integers $q_1, \dots, q_{3n}$ and h with

$$\sum_{i=1}^{3n} q_i = n \cdot h \quad \text{and} \quad \frac{h}{4} < q_i < \frac{h}{2} \quad \text{for } 1 \leq i \leq 3n$$

We want: yes/no, whether n disjoint triples of q_i's with sum h exist

Reduction: 3-partition $\longrightarrow$ DDDP

- Instance of 3partition:

$$Q := \{q_1, \dots, q_{3n}\} \text{ and } h \text{ (as above)}$$

- Define: $S := \sum_{i=1}^{3n} q_i = n \cdot h$

$t := (n+1) \cdot S$

$A' := Q \quad \hat{A} := \{\underbrace{2t, \dots, 2t}_{n-1 \text{ times}}\}$

$B' := \{\underbrace{h+2t, \dots, h+2t}_{n-2 \text{ times}}\} \quad \hat{B} := \{h+t, h+t\}$

$C' := Q \qquad \hat{C} := \underbrace{\{t, \dots, t\}}_{2n-2 \text{ times}}$

- Instance of DDDP:

$$A := A' \cup \hat{A}$$
$$B := B' \cup \hat{B}$$
$$C := C' \cup \hat{C}$$

$$\left[\sum_{a \in A} a = \sum_{b \in B} b = \sum_{c \in C} c = S + (2n-2)t \right]$$

"$\Longrightarrow$":

Assumption: solution for 3-partition exists $\rightarrow$ n disjoint triples with sum 1 (and q_i's (and thus a_i's and c_i's) already correspondingly ordered)

- arrangement of A (π_A): q_i's of each triple next to each other, separated by one $2t$-fragment

- arrangement of B (π_B): $h+t$-fragment, all $2t+h$-fragments, $h+t$-fragment

- arrangement of C (π_C): triples as in A, triples separated by two t-fragments

@ a): $Pos(\pi_C) = Pos(\pi_A) \cup Pos(\pi_B)$ by construction

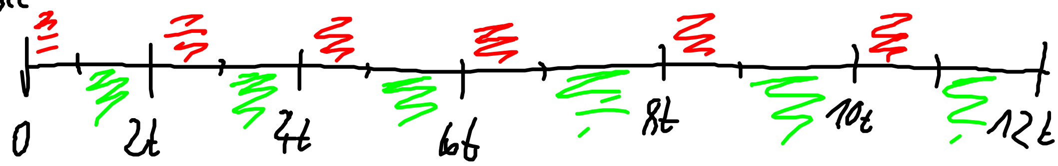
$\rightarrow Dist(Pos(\pi_A) \cup Pos(\pi_B)) = Dist(Pos(\pi_C)) = C$

@ b): position sets of A and B are disjoint (except for first/last val.)

1) every point in $Pos(\pi_A)$ is sum of value $< t$ and even multiple of t

2) every point in $Pos(\pi_B)$ is sum of multiple of h and odd multiple of t

example:



points from $Pos(\pi_A)$, since added value $< t$

points from $Pos(\pi_B)$, since at most $n-1$ multiples of h
$\rightarrow (n-1)\cdot h < S < t$

$\rightsquigarrow$ arrangements / permutations = solution to DDDP

" $\Longleftarrow$ "

Assumption: solution for DDDP exists $\rightarrow$ $Pos(\pi_A)$, $Pos(\pi_B)$, $Pos(\pi_C)$

Consider $Pos(\pi_B)$ and $Pos(\pi_C)$.

1) every point in $Pos(\pi_B)$ = sum of multiple of h and multiple of t
(all points differ in multiple of h)

2) $Pos(\pi_B) \subseteq Pos(\pi_C)$ holds

$\rightarrow$ $n+1$ points of the form from 1) must exist in C
(but $\hat{c}_j$'s contribute only t's)

$\rightarrow$ points in $Pos(\pi_C)$ must be positioned such that the c_i's contribute to multiple of h

$\rightarrow$ leads to n subsets of C' (each sums up to h)

Since $h/4 < q_i < \frac{h}{2}$ for $1 \le i \le 3n$, every subset must contain exactly 3 elements.

$\rightsquigarrow$ corresponding subsets of q_i's = solution to 3-partition

Problem 14

In the following: <u>not</u> the worst-case, but still exponential

Pattern (for $n \geq 3$):

$$A_n = \{[1, n-1], [2, n], [3, n-2], [4, n-3], \ldots, [i, n-i+1], \ldots, [n, 1]\}$$

"one too little" "one too much"

with $|A_n| = \sum_{i=1}^{n} i = \binom{n+1}{2} \rightsquigarrow A_n$ valid input

Idea:

- computation of 1 execution of Platziere (excluding recursive calls) takes polynomial time in n
  $\rightsquigarrow$ need exponentially many recursive calls
- each recursive call (without the follow-up recursive calls) adds one 'node' to the backtracking tree
  $\rightsquigarrow$ compute size of backtracking tree T_n resulting from A_n (size in terms of # inner nodes)

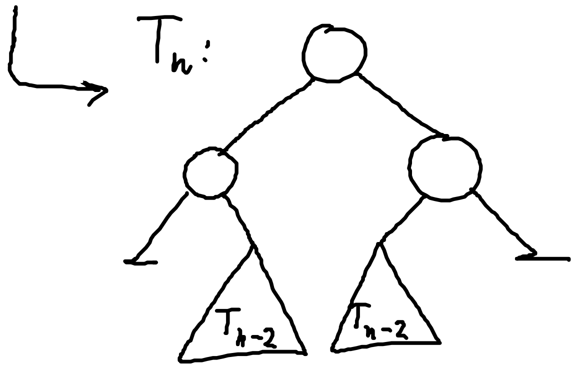
Assertion 1

- build trees $A_3, A_4, \ldots$ to verify first values of $|T_n|$

- for $n \geq 5$, "have a look at the trees and guess the pattern"

$\rightsquigarrow$ backtracking subtrees are isomorphic to full backtracking tree with smaller n

$$a_3 = 3$$
$$a_4 = 5$$
$$a_n = 2a_{n-2} + 3 \quad , n \geq 5$$



Assertion 2

again: "look at the numbers and guess a pattern"
+ proof by induction

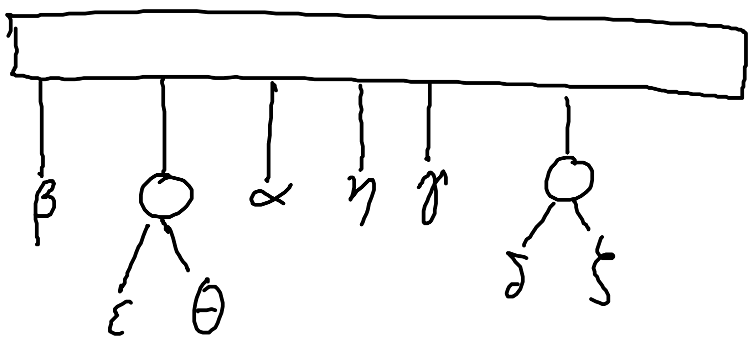
$$a_n = 2^{\lceil \frac{n}{2} \rceil - 2} \cdot b_n + 3 \sum_{j=0}^{\lceil \frac{n}{2} \rceil - 3} 2^j \quad \text{for } n \geq 3$$

$$\text{with } b_n = \begin{cases} 3, & n \text{ uneven} \\ 5, & n \text{ even} \end{cases} = 5 - 2(n \bmod 2)$$

$\rightsquigarrow$ $\Theta(\sqrt{2}^n)$ steps to decide that input A_n is infeasible

Problem 15

Result:



$\rightsquigarrow$ 8 permutation allowed