Exercise 5

Problem 13

DDDP

We have: multisets A, B, C with $S = \sum_{a\in A} a = \sum_{b\in B} b = \sum_{c\in C} c$

We want: yes/no, whether permutations π_A, π_B, π_C exist such that

(a) $C = Dist(Pos(\pi_A) \cup Pos(\pi_B))$

(b) $Pos(\pi_A) \cap Pos(\pi_B) = \{0, S\}$

3-partition

We have: integers $q_1, \dots, q_{3n}$ and h with

$$\sum_{i=1}^{3n} q_i = n \cdot h \quad \text{and} \quad \frac{h}{4} < q_i < \frac{h}{2} \quad \text{for } 1 \le i \le 3n$$

We want: yes/no, whether n disjoint triples of q_i's with sum h exist

Reduction: 3-partition ⟶ DDDP

- Instance of 3partition:

  $Q := \{q_1, \dots, q_{3n}\}$ and h (as above)

- Define: $S := \sum_{i=1}^{3n} q_i = n \cdot h$

  $t := (n+1) \cdot S$

  $A' := Q \quad \hat{A} := \underbrace{\{2t, \dots, 2t\}}_{n-1 \text{ times}}$

  $B' := \underbrace{\{h+2t, \dots, h+2t\}}_{n-2 \text{ times}} \quad \hat{B} := \{h+t, h+t\}$

$C' := Q \qquad \hat{C} := \underbrace{\{t, \dots, t\}}_{2n-2 \text{ times}}$

- Instance of DDDP:

$$A := A' \cup \hat{A}$$
$$B := B' \cup \hat{B}$$
$$C := C' \cup \hat{C}$$

$$\left[\sum_{a \in A} a = \sum_{b \in B} b = \sum_{c \in C} c = S + (2n-2)t \right]$$

$\Longrightarrow$:

Assumption: solution for 3-partition exists $\longrightarrow$ n disjoint triples with sum

(and q_i's (and thus a_i's and c_i's) already correspondingly ordered)

- arrangement of A (π_A): q_i's of each triple next to each other, separated by one $2t$-fragment
- arrangement of B (π_B): $h+t$-fragment, all $2t+h$-fragments, $h+t$-fragment
- arrangement of C (π_C): triples as in A, triples separated by two t-frag.-

@ a): $Pos(\pi_C) = Pos(\pi_A) \cup Pos(\pi_B)$ by construction

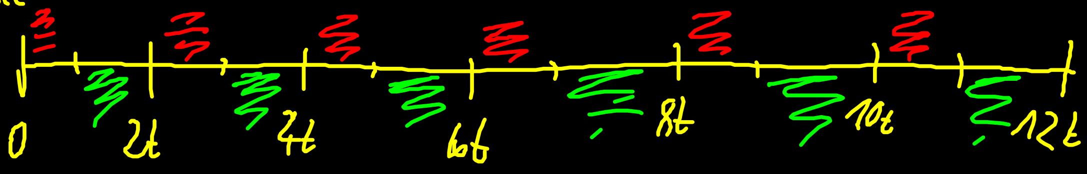
$\longrightarrow Dist(Pos(\pi_A) \cup Pos(\pi_B)) = Dist(Pos(\pi_C)) = C$

@ b): position sets of A and B are disjoint (except for first/last val.)

1) every point in $Pos(\pi_A)$ is sum of value $< t$ and even multiple of t

2) every point in $Pos(\pi_B)$ is sum of multiple of h and odd multiple of t

example:



🟥 points from Pos(π_A), since added value $< t$

🟩 points from Pos(π_B), since at most $n-1$ multiples of h

$\rightarrow (n-1)\cdot h < S < t$

⟶ arrangements / permutations = solution to DDDP

"$\Longleftarrow$"

Assumption: solution for DDDP exists ⟶ Pos(π_A), Pos(π_B), Pos(π_C)

Consider Pos(π_B) and Pos(π_C).

1) every point in Pos(π_B) = sum of multiple of h and multiple of t (all points differ in multiple of h)

2) Pos(π_B) $\subseteq$ Pos(π_C) holds

⟶ $n+1$ points of the form from 1) must exist in C'

(but $\hat{c}_j$'s contribute only t's)

⟶ points in Pos(π_C) must be positioned such that the c_i's contribute to multiple of h

⟶ leads to n subsets of C' (each sums up to h)

Since $h/4 < q_i < \frac{h}{2}$ for $1 \le i \le 3n$, every subset must contain exactly 3 elements.

$\rightsquigarrow$ corresponding subsets of q_i's = solution to 3-partition

Problem 14

In the following: <u>not</u> the worst-case, but still exponential!

Pattern (for $n \geq 3$):

$$A_n = \{\underset{\substack{\uparrow \\ \text{“one too} \\ \text{little”}}}{[1, n-1]}, \underset{\substack{\uparrow \\ \text{“one too} \\ \text{much”}}}{[2, n]}, [3, n-2], [4, n-3], \dots, [i, n-i+1], \dots, [n, 1]\}$$

with $|A_n| = \sum_{i=1}^{n} i = \binom{n+1}{2}$ $\longrightarrow$ A_n valid input

Idea:

- computation of 1 execution of Platziere (excluding recursive calls) takes polynomial time in n
  $\longrightarrow$ need exponentially many recursive calls
- each recursive call (without the follow-up recursive calls) adds one 'node' to the backtracking tree
  $\longrightarrow$ compute size of backtracking tree T_n resulting from A_n
  (size in terms of # inner nodes)

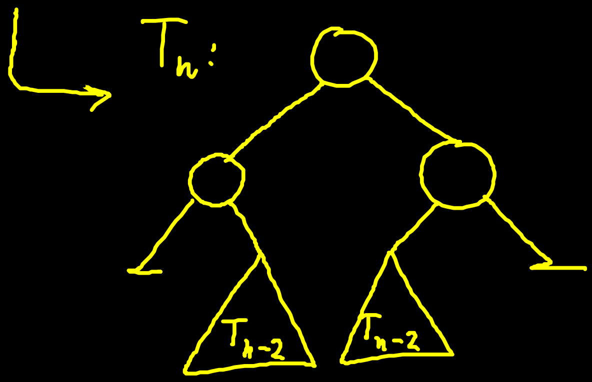
Assertion 1

- build trees A_3, A_4, … to verify first values of $|T_n|$

- for $n \geq 5$, "have a look at the trees and guess the pattern"

⇝ backtracking subtrees are isomorphic to full backtracking tree with smaller n

$$a_3 = 3$$
$$a_4 = 5$$
$$a_n = 2a_{n-2} + 3 \quad , n \geq 5$$



Assertion 2

again: "look at the numbers and guess a pattern"
+ proof by induction

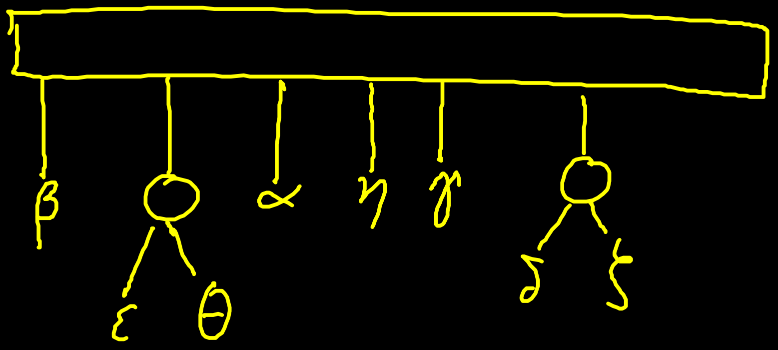
$$a_n = 2^{\lceil \frac{n}{2} \rceil - 2} \cdot b_n + 3 \sum_{j=0}^{\lceil \frac{n}{2} \rceil - 3} 2^j \quad \text{for } n \geq 3$$

$$\text{with } b_n = \begin{cases} 3, & n \text{ uneven} \\ 5, & n \text{ even} \end{cases} = 5 - 2(n \bmod 2)$$

⇝ $\Theta(\sqrt{2}^n)$ steps to decide that input A_n is infeasible

Problem 15

Result:



⟶ 8 permutation allowed