Alternative Procedure

Idea: Do not use a single cycle to cover all vertices but a set of cycles
$\rightsquigarrow$ cycle-cover for which each vertex must be part of exactly one cycle.

Here: Cycle-cover of minimal cost (according to distance graph).

Remark: A minimal cycle-cover can be computed in time $\mathcal{O}(n^3)$ (whereas the restriction to edges to be covered by at least one cycle leads to an $\mathcal{NP}$-complete problem).

Approxiamtion: Construct a single cycle (solution to TSP) from the minimal cycle-cover.

Steps:

1. Identify each cycle $c \in \mathcal{C}$ by one of its vertices $\rightsquigarrow$ set of vertices R.
2. Compute a minimal cycle-cover for the subgraph induced by $R \rightsquigarrow$ set of cycles $\mathcal{C}'$.
3. For each $c \in \mathcal{C}'$ delete the edge corresponding to a merge of minimal overlap $\rightsquigarrow$ broken cycles are considered as merged strings.
4. Concatenate these strings and expand each of its elements by the cycle from $\mathcal{C}$ it represents.

$K_1 = a_1 \to a_2 \to a_3 \to a_4 \to a_5 \to a_1$

$K_2 = b_1 \to b_2 \to b_3 \to b_4 \to b_5 \to b_6 \to b_1$

$K_3 = c_1 \to c_2 \to c_3 \to c_4 \to c_1$

$K_4 = d_1 \to \dots \to d_5 \to d_1$

$K_5 = e_1 \to \dots \to e_5 \to e_1$

$$U_6 = f_1 \to f_2 \to f_1$$
$$U_7 = g_1 \to g_2 \to g_3 \to g_1$$

$$S = \{a_1, \dots, a_4, b_1, \dots, b_6, \dots\}$$

$$K_1' = a_1 \to b_1 \to c_1 \to a_1 \rightsquigarrow \underbrace{\langle b_1, c_1, a_1 \rangle}_{u_1}$$
$$K_2' = d_1 \to e_1 \to d_1 \rightsquigarrow \underbrace{\langle e_1, d_1 \rangle}_{u_2}$$
$$K_3' = f_1 \to g_1 \to f_1 \rightsquigarrow \underbrace{\langle f_1, g_1 \rangle}_{u_3}$$

$$\Longrightarrow w' = u_1 u_2 u_3$$

expand

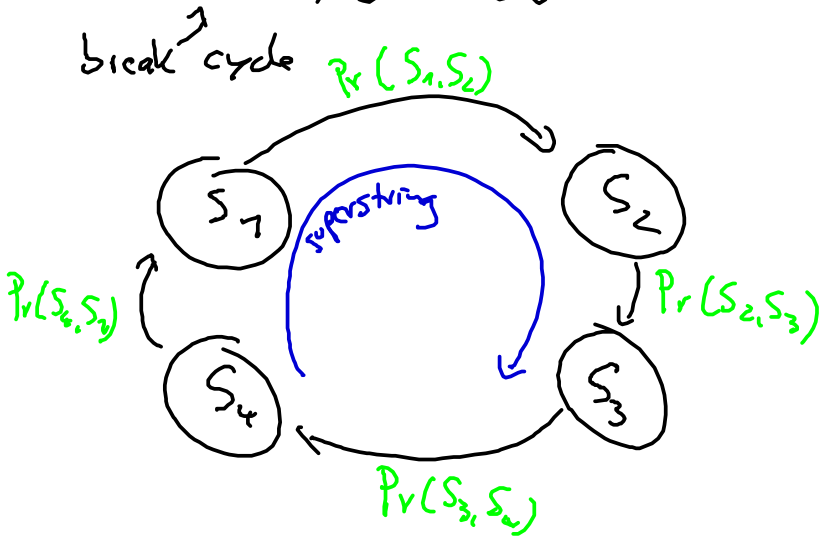
$$w = \langle \langle b_1, b_2, \dots, b_6, b_1 \rangle, \langle c_1, c_2, \dots, c_4, c_1 \rangle, \langle a_1, a_2, \dots, a_4, a_1 \rangle \rangle$$
$$\cdot \langle \langle e_1, \dots, e_5, e_1 \rangle, \langle d_1, \dots, d_5, d_1 \rangle \rangle$$
$$\cdot \langle \langle f_1, f_2, f_1 \rangle \langle g_1, g_2, g_3, g_1 \rangle \rangle$$

Theorem

The algorithm just outlined is a 3-approximation algorithm for SCSP.

Proof: Solution to TSP on digraph

⟶ solution to SCSP



cost for TSP: $pr(S_1,S_2)+pr(S_2,S_3)+pr(S_3,S_4)+pr(S_4,S_1)$ ⟶ $pr(S_4,S_1)+pr(S_1,S_2)+pr(S_2,S_3)+|S_3|$

Since $pr(S_i,S_j) < |S_i| \Rightarrow$ cost of TSP lower bound for cost SCSP.

Minimal cycle cover has cost at most the same as TSP ⟶ min. cycle cover lower bound for TSP.

$$\Longrightarrow cost(\mathcal{C}) = Opt_{cc}(DG(S)) \leq OPT_{SCS}(S)$$

$$cost(\mathcal{C}') = Opt_{cc}(DR(R)) \leq OPT_{SCS}(R)$$

since DG(R) is a subgraph of DG(S)

$\Longrightarrow \operatorname{cost}(e') \le \operatorname{cost}(e)$ (o)

minimal overlap of cycle K_i'

Observation: $|U_i| = \operatorname{cost}(K_i') + \operatorname{minov}(K_i')$

$$\Longrightarrow |w'| \le \operatorname{cost}(e') + \sum_{c' \in e'} \operatorname{minov}(c') \quad (*)$$

Replacing within w' all representatives by their cycle:

gives us, e.g. $\underbrace{\operatorname{Pref}(a_1,a_2)\cdot\operatorname{Pref}(a_2,a_3)\cdot\operatorname{Pref}(a_3,a_4)\cdot\operatorname{Pref}(a_4,a_1)}_{\text{cost of cycle}}\cdot \underline{\underline{a_1}}$

$\Longrightarrow$ w' „gets longer" by the cost of resp. cycle.

$$\Longrightarrow |w| \le |w'| + \operatorname{cost}(e)$$

$$\overset{(*)}{\le} \operatorname{cost}(e') + \sum_{c' \in e'} \operatorname{minov}(c') + \operatorname{cost}(e)$$

$$\overset{(o)}{\le} 2\cdot\operatorname{cost}(e) + \sum_{c' \in e'} \operatorname{minov}(c')$$

We obviously have

$$\sum_{c' \in \mathcal{C}'} \text{minov}(c') \leq \frac{1}{2} \cdot \sum_{c' \in \mathcal{C}'} \sum_{(s_i, s_j) \in c'} ov(s_i, s_j)$$

(every cycle at least 2 edges)

$$\Longrightarrow |w| \leq 2 \cdot \text{cost}(\mathcal{C}) + \frac{1}{2} \cdot \sum_{c' \in \mathcal{C}'} \sum_{(s_i, s_j) \in c'} ov(s_i, s_j)$$

Lemma: Let c_1, c_2 be two cycles of a minimal cycle cover $\mathcal{C}$ and $s_1 \in c_1$, $s_2 \in c_2$ to „strings' of those cycles. Then we have

$$ov(s_1, s_2) < \text{cost}(c_1) + \text{cost}(c_2).$$

Lemma $\Longrightarrow$ $|w| \leq 2 \cdot \text{cost}(\mathcal{C}) + \frac{1}{2} \cdot \underbrace{\sum_{c' \in \mathcal{C}'} \sum_{(s_i, s_j) \in c'} \left(\text{cost}_{\mathcal{C}}(\text{cycle}(s_i)) + \text{cost}_{\mathcal{C}}(\text{cycle}(s_j)) \right)}_{\text{cost}(\mathcal{C})}$

$$= 3 \cdot cost(e)$$

$$\leq 3 \cdot Opt_{scs}(S)$$

$\square$