String Algorithms

String-Matching:

Definition

Given a text $T \in \Sigma^\star$ and a string $P \in \Sigma^+$, the string matching problem *is to determine all $s \in \mathbb{N}_0$, satisfying:*

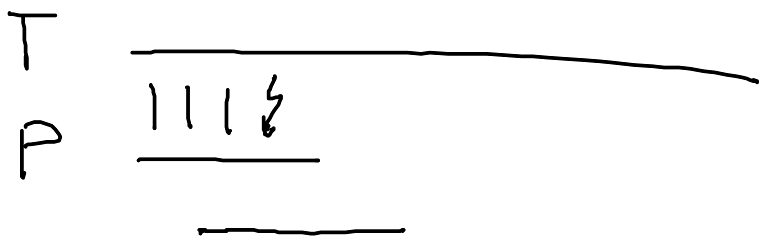
$$(\exists v \in \Sigma^s, w \in \Sigma^\star)(T = vPw).$$

The number s from this definition is named *shift*.
A shift is called *feasible*, if P is found at the respective place in T, otherwise s is called *infeasible*.

Naïve algorithm: Try all shifts $s \in [0, |T| - |P|]$ one by one.

Worst case running time: $\mathcal{O}(|P| \cdot |T|)$. E.g. if $P = a^m$, $T = a^n$, $m, n \in \mathbb{N}$, $m < n$.

Reason of the slow running time: Knowledge about T gained in previous steps is not used. If e.g. $P = aaab$ and $s = 0$ is a feasible shift, we already know that $s = 1$, $s = 2$ and $s = 3$ are infeasible.
Thus algorithm is implemented in Java runtime library!

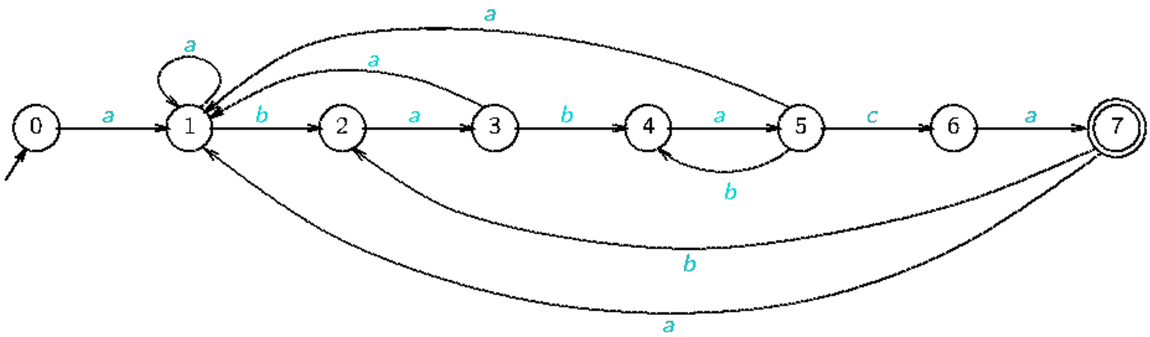


Efficient String Matching Algorithms

Here we only give an overview:

1) Using finite automata

Example: $P = ababaca$ and $T = abababacaba$.



Fundamental definition:

Definition

The suffix function $\sigma_P : \Sigma^* \to \{0, 1, \ldots, |P|\}$ of P *is defined by*

$$\sigma_P(X) := \max_{k \in \mathbb{N}_0}\{k \mid P_{0,k} \sqsupset X\},$$

i.e. $\sigma_P(X)$ *is the length of the longest prefix of* P *being a suffix of* X *(where* $P \sqsupset X$ *denotes that* P *is a suffix of* X *and* $P_{0,k}$ *is the length* k *prefix of* P*).*

Now with $\delta(q,a) := \sigma_P(P_{0,q}, a)$, $\forall q \in Q$ and $\forall a \in \Sigma$ a linear scan of the text is sufficient to find all feasible shifts.
Preprocessing: $\mathcal{O}(m^3 \cdot |\Sigma|)$-algorithm to compute δ:

```
m:=|P|;
for q:=0 to m do begin
  for a in Sigma do begin
    k:=min(m+1,q+2); // P[0..k] should be
                     // suffix of P[0..q]+a
    repeat
      k:=k-1;
    until (P[0..k] is suffix of (P[0..q]+a));
    delta[q,a]:=k;
  end;
end;
```

- `P[i..j]` denotes substring $P_{i,j}$ of P,
- operator + on strings describes concatenation.

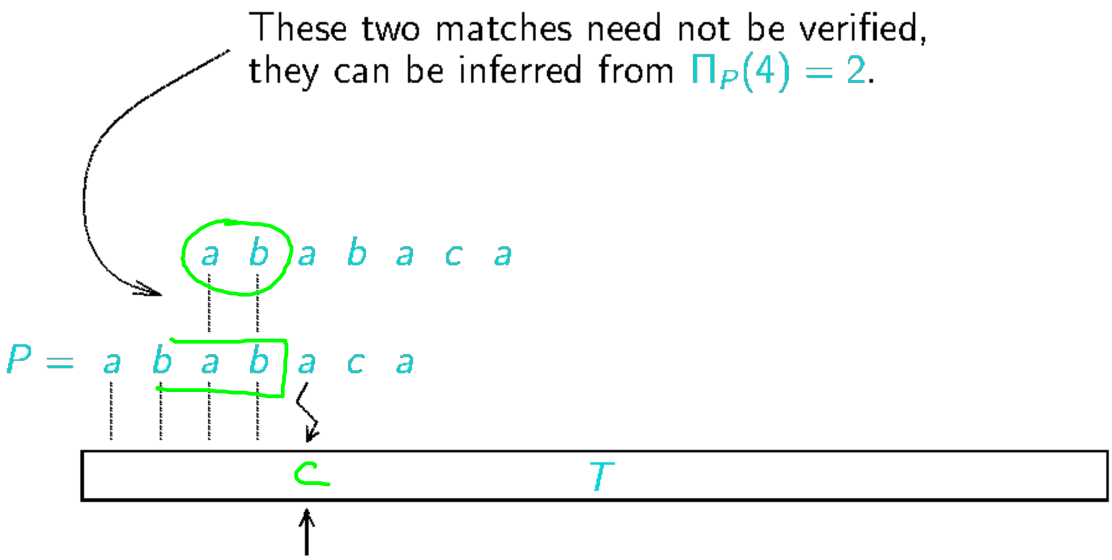
2) Knuth-Morris-Pratt Algorithm (KMP)

Definition

Let $P \in \Sigma^m$ *a string. The* prefix function
$\Pi_P : \{1,2,\ldots,m\} \to \{0,1,\ldots,m-1\}$ of P *is defined by*

$$\Pi_P(q) := \max_{k \in \mathbb{N}_0}\{k \mid k < q \wedge P_{0,k} \sqsupset P_{0,q}\}.$$

Example: For $P = ababaca$, $\Pi_P(4) = 2$ holds, since $k = 2$ is the maximum value for which $P_{0,k} \sqsupset P_{0,4}$, $k < 4$, holds. This leads to the following situation:



```
1     m:=|P|;
2      Pi[1]:=0;
3      k:=0;
4      for q:=2 to m do begin
5          while (k>0) and (P[k+1]<>P[q]) do
6              k:=Pi[k];
7          if P[k+1]=P[q] then
8              k:=k+1;
9          Pi[q]:=k;
10     end;
```

Running time: (amortized analysis using the potential method)

- Let the i-th operation Op_i be the i-th iteration of the **for**-loop. (Executing lines 7 through 9 yields constant cost c.).
- $\Rightarrow$ Cost C_i of Op_i is c plus number of iterations of the **while**-loop.
- **while**-loop iterated often only if k is large. (Assignment in line 6 strictly decreasing). **while**-loop iterated often leaves k small.

Hence we choose $\mathrm{pot}(i) = k$.

Amortized cost

$$C_i + \overbrace{\mathrm{pot}(i) - \mathrm{pot}(i-1)}^{\text{increase of potential during } \mathrm{Op}_i} .$$

To reach j iterations of the **while**-loop, $k \geq j$ is required.
$\rightsquigarrow \geq j$ previous operations need to have gone without decreasing k during the **while**-loop but increasing k by 1 in line 8.

These operations have actual cost c, but are accounted with cost $c+1$ in our analysis.
($\rightsquigarrow$ Overcharging of j to account for the cost of j iterations of the **while**-loop).

On the other hand $C_i = c + j$ holds for the iteration, however the *increase of potential* is $-j$ (k is reduced by j, thus $\text{pot}(i) - \text{pot}(i-1) = -j$) resp. $-j+1$, if line 8 is evaluated after the loop.
Hence amortized costs are $\leq c + j - j + 1 = c + 1$. (Here the previous overcharging and the cost of the **while**-loop are balanced, because in amortized analysis an operation including iterations of the **while**-loop is also rated with $c + 1$ at most.)
Our discussion therefor leads to

$$C_i + \text{pot}(i) - \text{pot}(i-1) \leq c + 1 = \mathcal{O}(1).$$

Summing the amortized costs of all iterations of the **for**-loop, we get

$$\sum_{2 \leq i \leq m} \overbrace{(C_i + \text{pot}(i) - \text{pot}(i-1))}^{\mathcal{O}(1)} = \text{total cost} + \text{pot}(m) - \text{pot}(1).$$

Hence: $\text{pot}(m) - \text{pot}(1) \geq 0 \Rightarrow$ Summed amortized costs are upper bound of actual costs. This requirement is however fulfilled trivially as k never gets negative and starts with 0 in line 3.
$\Rightarrow$ Upper bound of

$$(m-1) \cdot \mathcal{O}(1) = \underline{\mathcal{O}(m)}$$

for the running time of our algorithm to compute the prefix function.

Knuth-Morris-Pratt (KMP) algorithm

```
1   n:=|T|;
2   m:=|P|;
3   // Compute prefix function Pi here
4   q:=0;
5   for i:=1 to n do begin
6       while (q>0) and (P[q+1]<>T[i]) do
7           q:=Pi[q];
8       if P[q+1]=T[i] then q:=q+1;
9       if q=m then do begin
10          print('Occurrence at shift ',i-m);
11          q:=Pi[q];
12      end;
13  end;
```

Remarks:

- KMP has (optimal) running time in $\mathcal{O}(m+n)$ which can be proven by a similar analysis.
- The knowledge of Π_P makes it possible do compute δ of $SMA(P)$ in linear time.
- Comparing the naïve method and the (optimized) KMP algorithm by dividing the expected number of comparisons both algorithms need on random texts we find

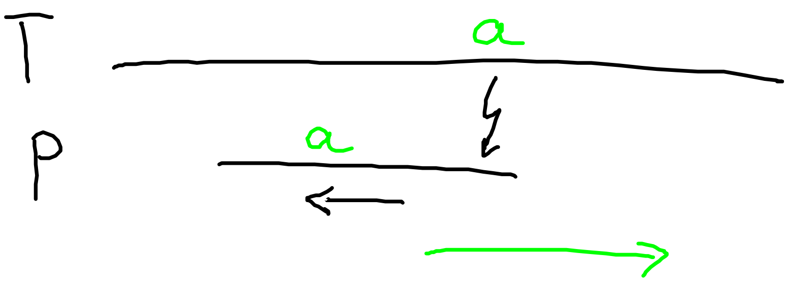
$$\mathsf{KMP/NAIVE} = 1 - \frac{1}{c} + \frac{1}{c^2} + \frac{c-1}{c^m}.$$

So if m and c are large enough both methods are almost equal.

3) The Boyer-Moore algorithm

Application: P long, Σ relatively large.

- **Core:** Naïve method: By setting `s:=s+1` in lines 12 and 14 we get an implementation of the naïve method.
- **Notable:** P is compared to the text from right to left.
- **Speed-up:** In case of a mismatch two heuristics (*bad character* heuristic (`lambda`), *good-suffix* heuristic (`gamma`)) give an increment for s which does not miss a feasible shift and is usually greater than 1.



Worst case running time of the Boyer-Moore algorithm is in $\mathcal{O}((|T| - |P| + 1) \cdot |P| + |\Sigma|)$ (and usually in $\Theta((|T| - |P| + 1) \cdot |P|))$, as

- the computation of `lambda` takes $\mathcal{O}(|P| + |\Sigma|)$ time,
- the computation of `gamma` takes $\Theta(|P|)$ time and
- the algorithm does not use more than $\Theta(|P|)$ time on each of the at worst $|T| - |P| + 1$ shifts.

Practise: BM often the best choice as the worst case rarely occurs and the two heuristics give relatively large increments on the considered shifts. $\Rightarrow$ „sublinear" (in in length of text) running time. BM *faster* than optimized KMP algorithm.

4) Boyer-Moore-Horspool algorithm

Variation of BM with only one heuristic similar to the bad-character heuristic. (Negative movement is avoided.)
Mismatch on comparing P with $T_{i-|P|+1,i} \Rightarrow P$ is moved to the right by $d(T_i)$ positions, where

$$d(x) := \min_{1 \le k \le |P|} \{k \mid k = |P| \vee P_{|P|-k} = x\}.$$

Intuition: T_i is brought to a match with a character of P (if possible). The minimizing guarantees that no potentially feasible shift is omitted.

Running time: Worst case $\Theta(|T| \cdot |P|)$, average case (sub)linear. The constant of the linear term in the average running time is asymptotical ($|T| \to \infty$)

$$\frac{1}{|\Sigma|} + \mathcal{O}\left(\frac{1}{|\Sigma|^2}\right).$$

5)Karp-Rabin algorithm

6)Algorithm of Aho and Corasick

This algorithm finds all occurrences of a set of search terms in a text (*set matching problem*) at the same time. This is achieved by organising the strings in a *search term tree*, a directed tree satisfying the following conditions:

- Each edge is labeled with a symbol from Σ.
- Edges leaving the same node are labeled with different symbols.
- For each search term w there is exactly one node such that the path from the root to this node is labeled with w.
- Each leave is associated with a search term.

Searching in the text:

- Traverse the search term tree according to the letters of T.
- Reaching a node corresponding to a search term means we have found this term.
- If no outgoing vertex for the next symbol exists:
  $\Rightarrow$ *failure links*: Link from node v to node w such that a path from the root to w is equal to the longest suffix of the path from the root to v.
- Determining these links: Refer to $SMA(P)$, the search term tree is like a string matching automaton for a set of strings.
- **Difference:** failure links are not associated with symbols from the alphabet.
- Traversing a failure link does not consume a symbol of the text, but increase the current shift by the number of levels we went up in the tree.
- It is possible that multiple failure links are traversed in direct succession.
- If the current node is the root and there is no matching edge we stay at the root and advance to the next symbol of the text.

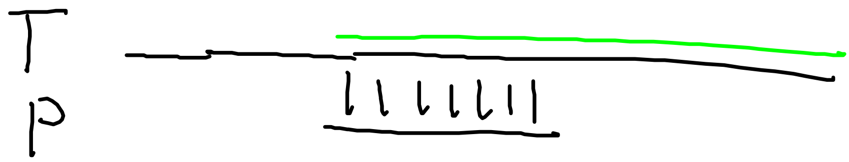
Suffix Trees

Idea: P appears in T, if and only if P is a prefix of a suffix of T.

Definition

Let $T \in \Sigma^n$ a text. A directed tree $B_T = (V, E)$ with root r is called a simple suffix tree for T, *if it satisfies the following conditions:*

1. *B_T has exactly n leaves labeled with numbers 1 to n.*
2. *Every edge in B_T is labeled with a symbol from Σ.*
3. *All edges leaving an (internal) node are labeled differently.*
4. *The path from r to leaf i is labeled with $T_{i,n}$.*



Method: Construction of a simple suffix tree B_T.
Input: Text $T \in \Sigma^n$.

Step 1: Let $T' = T \cdot \$$, $\$ \notin \Sigma$; let $\Sigma' = \Sigma \cup \{\$\}$.
Step 2: Initialize B_T with childless root r.
Step 3: For i from 1 to n repeat:

- Traverse B_T starting at r along the path $T_{i,n} \cdot \$$ until node x, reached by symbol T_k, has no leaving edge matching T_{k+1}.
- Append to x a linear list of nodes, the corresponding edges labeled with $T_{k+1,n} \cdot \$$.
- Label the new leaf with i.

String-Matching: Deciding with running time $\Theta(|P|)$.
Finding all matches: Additional effort proportional to the size of the subtree reached by P.

Problem: A simple suffix tree may have size in $\Omega(|T|^2 \cdot |\Sigma|)$.

Reason: Nodes with only one successor.

$\Rightarrow$ Allow each (nonempty) word as label and eliminate unary nodes. Words are represented by start- and end-position in the text.

Definition

Let $T \in \Sigma^n$ *a text. A directed tree* $B_T = (V, E)$ *with root* r *is called* compact suffix tree *for* T*, if it satisfies the following conditions:*

1. B_T *has exactly* n *leaves, labeled with numbers* 1 *to* n.
2. *Each internal node of* B_T *has at least two successors.*
3. *The edges of* B_T *are labeled with substrings of* T.
4. *Labels of edges leaving the same node start with pairwise different symbols.*
5. *The path from the root to leaf* i *is labeled with* $T_{i,n}$, $1 \leq i \leq n$.

Lemma

Let $T \in \Sigma^n$ *a text. A compact suffix tree* B_T *for* T *has* $\mathcal{O}(n)$ *nodes. Labeling all edges takes* $\mathcal{O}(n \log(n))$ *bits.*

Proof: • Every suffix tree has n leaves.

• internal nodes at least 2 successors

$\rightsquigarrow$ at most $n-1$ internal nodes.

$\Longrightarrow$ at most $2n-1 = \mathcal{O}(n)$ nodes

• $2n-1$ nodes in a tree have exactly $2n-2$ edges

• Each label (pair of positions in text) needs $\mathcal{O}(\log(n))$ bits

$\Longrightarrow \mathcal{O}(n \cdot \log(n))$ bits $\square$

Construction: (Ukkonen's algorithm)

Definition

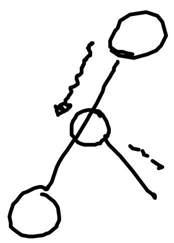
An implicit suffix tree *is the tree resulting from the compact suffix tree for $T \cdot \$$ by*

1. *removing all occurrences $\$$ from the labels.*
2. *removing unlabeled edges (and nodes which are afterwards no longer reachable from the root) and*
3. *removing nodes with only one child (merging the incoming and the outgoing edge to one edge labeled with the concatenation of the previous labels).*

Approach: Process T symbol by symbol from left to right (online algorithm) constructing implicit suffix trees IB_k, corresponding to the prefix $T_{0,k}$. IB_0 consists only of the root. IB_1 has two nodes (root and a leaf labeled with 1), connected by an edge labeled with $T_{1,1}$.

Now we construct IB_{i+1} from IB_i, $1 \leq i \leq n-1$ as follows:

```
for i := 1 to n − 1 do begin
    // Phase i + 1
    for j := 1 to i + 1 do begin
      Traverse IB_i along the path T[j..i];
      If necessary extend the tree at the
      position reached this way by T[i + 1];
      // Details follow
    end;
end;
```



Rule 1: If the path labeled $T_{j,i}$ ends in a leaf, T_{i+1} is appended to the label of the edge leading to the leaf.

Rule 2: If the path does **not** end in a leaf and there is no possibility to continue it with T_{i+1}, a new edge to a new leaf is created and labeled with T_{i+1}. The leaf is labeled j. If $T_{j,i}$ ends amidst an edge, additionally a new internal node has to be created at the respective position.

Rule 3: If the path can be continued with T_{i+1} nothing is done.

Example: $T = \underline{cba}cb$.

Caution: Nested loops + traversal along $T_{j,i}$ $\Rightarrow$ running time cubic in length of text.

Tricks:
1.) If for given j Rule 3 is applied for the first time:
$\Rightarrow$ The path labeled $T_{j,i}$ can be continued with T_{i+1}, created when inserting word w.
$\Rightarrow$ Suffixes of w inserted in an earlier phase guarantee existence of a continuation of $T_{j',i}$, $j' > j$ with T_{i+1}.

$\Rightarrow$ Rule 3 implies termination of the current phase.