7th Exercise Sheet for Kombinatorische Algorithmen, WS 14/15

Hand In: *Until Monday, 05.01.2015, 12:00,*
deliver or email to Raphael (`reitzig@cs.uni-kl.de`).

Problem 12

3 + 1 + 1 + 1 points

We consider the following refined bipartite matching problem.

> **Input:** Sets $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ of players with rankings of the individuals of opposite type.
>
> That is, for each a_i there is a total preference relation[1] $\prec_{a_i} \subseteq B \times B$, and likewise for every b_i, we have the relation $\prec_{b_i} \subseteq A \times A$.
>
> **Output:** A *Nash-matching* of A and B, that is a (bipartite) matching $M \subseteq A \times B$ for which no two players $a \in A$ and $b \in B$ fulfill the following property:
>
> There are $a' \in A$ and $b' \in B$ with $a \neq a'$ and $b \neq b'$,
>
> (NM 1) $(a, b'), (a', b) \in M$,
>
> (NM 2) $b \prec_a b'$ and
>
> (NM 3) $a \prec_b a'$.

We say that "c prefers x to y" if (and only if) $x \prec_c y$. Informally speaking, a Nash-matching is a matching where no two individuals have an incentive to leave their current matching partners in order to form a new pair.

a) Show that there always exists a Nash-matching of A and B by describing an algorithm for constructing such a matching.

   You may skip the running time analysis of your algorithm, but make sure you prove its correctness.

b) **Prove** or **disprove**:
   For any $n \geq 2$, there is a Nash-matching for certain preference relations that contains a pair (a, b), where a likes b *least* of all B-players and likewise b prefers all other A-players to a.

c) **Prove** or **disprove**:
   For any $n \geq 2$, there is a Nash-matching for certain preference relations that contains players $a \in A$ and $b \in B$ which are both paired with their *least* preferred partners, but they are *not* paired with each other.

d) **Prove** or **disprove**:
   For any $n \geq 2$, there is a Nash-matching for certain preference relations where *no* player is paired with its most preferred partner.

[1] The relations are total orders, i. e. any two elements b, b' are either equal or $b \prec_{a_i} b'$ or $b' \prec_{a_i} b$.