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Exercise Sheet 9 for Algorithm Engineering, SS 14

Hand In: *Until Monday, 30.06.2014, 10:00 am, email to* `wild@cs...` *or in lecture.*

Problem 21

3 points

Let $F(z) = \sum_{k\geq 1} d(k)e^{-kz}$, where $d(k)$ is the number of divisors of k.
Expand $F(z)$ around $z = 0$ up to the term $\mathcal{O}(z)$.

Hint: The *zeta function* $\zeta(s)$ is the *Dirichlet generating function* for the series $1, 1, 1, \ldots$ Recall the convolution formula for two Dirichlet generating functions $A(z) = \sum_{n\geq 1} a_n n^{-z}$ and $B(z) = \sum_{n\geq 1} b_n n^{-z}$:

$$A(z)B(z) \;=\; \sum_{n\geq 1} h_n n^{-z}, \qquad \text{with} \qquad h_n \;=\; \sum_{d|n} a_d b_{n/d}.$$

Problem 22

6 points

Let I_n be the (random) number of *inversions* of a permutation of $1, \ldots, n$ drawn uniformly at random from all $n!$ permutations.

Show that

$$I_n \;\overset{\mathcal{D}}{=}\; U_0 + U_1 + \cdots + U_{n-1}\,,$$

where $X \overset{\mathcal{D}}{=} Y$ means that X and Y have the same distribution and where the U_i are all independent and drawn uniformly from $\{0, \ldots, i\}$ for $0 \leq i < n$.

Moreover, derive general formulas in n for

a) the minimal and
b) the maximal value of I_n, as well as
c) the expected value $\mathbb{E}[I_n]$
d) and the variance $\mathbb{V}[I_n]$.
e) What does the result tell us about sorting algorithms that only compare neighboring elements?