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Exercise Sheet 5 for Algorithm Engineering, SS 14

Hand In: *Until Monday, 26.05.2014, 10:00 am, email to* `wild@cs...` *or in lecture.*

Problem 10

4 points

A binary tree is called Δ-*height-balanced*, $\Delta \in \mathbb{N}$, if for each of its nodes the difference of the heights of its subtrees is at most Δ. (1-height-balanced trees are known as AVL-trees.)

Show: The maximum height $h_{\max}$ of a Δ-height-balanced tree with n nodes satisfies:

$$h_{\max} \sim C_\Delta \operatorname{ld}(n), \qquad n \to \infty,$$

where C_Δ is a constant depending on Δ.

Hint: One way to start is to determine the generating function for the sequence of worst-case tree heights and a sufficiently small interval containing its dominant singularity.

Problem 11

3 + 1 points

In this exercise, we prove the famous *binomial theorem*. Recall the definition of binomial coefficients for $r \in \mathbb{C}$ and $k \in \mathbb{Z}$:[1]

$$\binom{r}{k} := \begin{cases} \dfrac{r^{\underline{k}}}{k!}, & \text{for } k \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

By $r^{\underline{k}}$, we denote the falling factorial *"r to the k falling"*, which is recursively defined by $r^{\underline{0}} = 1$ and $r^{\underline{k+1}} = r \cdot (r-1)^{\underline{k}}$ for $k \in \mathbb{N}$.

[1] There is an even more general definition relying on the *Gamma function* to generalize the notion of factorials to arbitrary complex k. For the binomial theorem, however, nonintegral k do not occur and it is more convenient to stick to the given elementary definition.

a) Let $z \in \mathbb{C}$ and assume that at least on of the following two conditions holds:
   (i) $|z| < 1$ or
   (ii) $r \in \mathbb{N}$.

   Show that then

$$(1+z)^r \;=\; \sum_k \binom{r}{k} z^k.$$

   Hint: Consider the Taylor series expansion of $f(z) := (1+z)^r$ around $z = 0$.

   You may use $\binom{r}{n} \in \mathcal{O}(n^{-r-1})$ as $n \to \infty$ for constant $r \in \mathbb{C}$ without proof.

b) Assume now that $x, y \in \mathbb{C}$ and at least one of the following conditions holds:
   (i) $y \neq 0$ and $|x/y| < 1$ or
   (ii) $r \in \mathbb{N}$.

   Show that then

$$(x+y)^r \;=\; \sum_k \binom{r}{k} x^k y^{r-k}.$$

   Hint: Use a).

Problem 12

4 + 3 + 2 points

In this exercise, we see by example how we can compute coefficients of "simple" algebraic functions *precisely* and how to obtain concise asymptotics for them. Generalizing this approach leads to a proof of *Darboux's theorem*.

a) Let the generating function $C(z) = \sum_{n \geq 0} c_n z^n$ be given by

$$C(z) \;:=\; \frac{1-\sqrt{1-4z}}{2z} \;=\; \frac{1}{2z} \;-\; \left(1-4z\right)^{1/2} \cdot \frac{1}{2z}\,. \qquad (2)$$

   Show that then holds

$$c_n \;=\; -\tfrac{1}{2} \cdot 4^{n+1} \binom{n-\frac{1}{2}}{n+1} \;=\; \frac{1}{n+1}\binom{2n}{n}\,.$$

   (You are expected to prove both equalities, but partial credit is given to solutions deriving only the first one.)

   Hint: Use the binomial theorem.

   Hint: There are numerous identities for manipulating binomials, some of which might come in handy here (see, e. g., the TCS cheat sheet).

b) Show that for $n \in \mathbb{N}$ and $c \in \mathbb{C}$ with $-r \notin \mathbb{N}$ holds

$$\binom{n+c}{n} \sim \frac{n^c}{\Gamma(c+1)}, \qquad (n \to \infty).$$

Partial credit is given to solutions for $c \in \mathbb{Z}$ only.

c) Prove the asymptotic

$$[z^n](1 - \tfrac{z}{\rho})^{-\omega} \sim \frac{1}{\Gamma(\omega)} \cdot n^{\omega-1}\rho^{-n}$$

for $\rho > 0$ and $-\omega \notin \mathbb{N}$.