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Exercise Sheet 7 for Combinatorial Algorithms, SS 13

Hand In: *Until Monday, 17.06.2013, 12:00,*
box in the group's hallway or email to `wild@cs.uni...`.

Problem 11

$3 + 1 + 1 + 1$ points

We consider the following refined bipartite matching problem.

Given two sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ of players with rankings of the individuals of opposite type, i. e. for each a_i, there is a total preference relation[1] $\prec_{a_i} \subseteq B \times B$, and likewise for every b_i, we have the relation $\prec_{b_i} \subseteq A \times A$. We say that "$c$ prefers x to y" iff $x \prec_c y$.

A (bipartite) matching $M \subseteq A \times B$ of A and B is a *Nash-matching*, when there are no two players $a \in A$ and $b \in B$ fulfilling (all of) the following properties:

(NM 1) M matches a with $b' \in B$ and b with $a' \in A$, where $a \neq a'$ and $b \neq b'$.

(NM 2) $b \prec_a b'$.

(NM 3) $a \prec_b a'$.

Informally speaking, a Nash-matching is a matching where no two individuals have an incentive to leave their current matching partners in order to form a new pair.

a) Show that there always exists a Nash-matching of A and B by describing an algorithm for constructing such a matching.

   You may skip the running time analysis of your algorithm, but make sure you prove its correctness.

b) **Prove** or **disprove**:
   For any $n \geq 2$, there is a Nash-matching for certain preference relations that contains a pair (a, b), where a likes b *least* of all B-players and likewise b prefers all other A-players to a.

[1] The relations are total orders, i. e. any two elements b, b' are either equal or $b \prec_{a_i} b'$ or $b' \prec_{a_i} b$.

c) **Prove** or **disprove**:
For any $n \geq 2$, there is a Nash-matching for certain preference relations that contains players $a \in A$ and $b \in B$, which are both paired with their *least* preferred partners, but they are *not* paired with each other.

d) **Prove** or **disprove**:
For any $n \geq 2$, there is a Nash-matching for certain preference relations where *no* player is paired with its most preferred partner.